Math 18500 Week 6: Fourier Series

Introduction to Fourier Series. We have seen how to solve equations of the form

$$my'' + by' + ky = \cos(\omega t).$$

Aside from the exceptional case $b = 0$, the general solution always takes the form

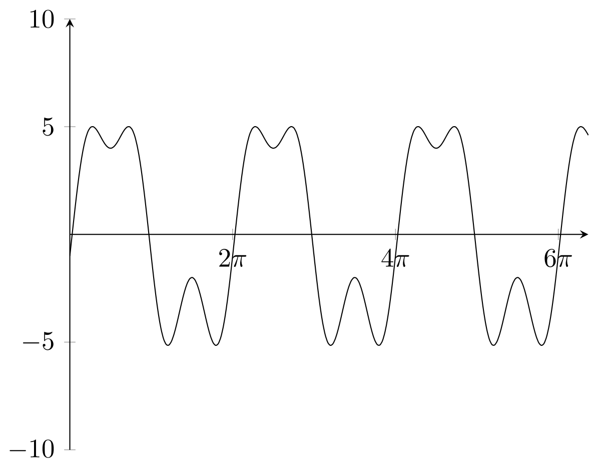
$$y = A\cos(\omega t - \phi) + y_h$$

where y_h is a general solution of the corresponding homogeneous equation.

More generally, we would like to be able to solve equations

$$my'' + by' + ky = f(t)$$

where the function $f(t)$ oscillates with some frequency, but is not necessarily a perfect sinusoid. A function $f(t)$ like this is said to be *periodic*. A typical example of a periodic function looks like this:



More formally, a function $f(t)$ is said to be periodic with period T (or T-periodic) if it satisfies the identity

$$f(t+T) = f(t)$$

Any function with period T also has period $2T$, $3T$, etc. - the *smallest* period of a function is called its *fundamental period*. Usually we will be thinking about periodic functions with period 2π, so

$$f(t+2\pi) = f(t).$$

We won't lose any generality by doing this - periodic functions with arbitrary periods can be converted to periodic functions with period 2π by "rescaling" the time variable. For instance, the function

$$g(t) = \cos(2\pi t)$$

is 1-periodic, but if we replace t with $\frac{u}{2\pi}$ we obtain the function

$$f(u) = g\left(\frac{u}{2\pi}\right) = \cos\left(2\pi \cdot \frac{u}{2\pi}\right) = \cos(u)$$

which is 2π-periodic.

One way to think about periodic functions is to listen to what they "sound like", by giving them as an input to a speaker and playing them like music. One example of a tool which allows you to do this on your computer can be found here:

https://synthtech.com/waveedit/

It's interesting to try out a few of the preset functions. If you listen to a pure sine wave, then you will hear a pure pitch (which actually sounds kind of strange!). If you listen to any other function you will hear a "superposition" of different pitches. In fact, the software will allow you to explicitly add and subtract pitches at will (by changing the heights of the bars below the graph of the function).

Mathematically, this idea can be expressed by writing $f(t)$ as a sum of pure sinusoids of different amplitudes and frequencies:

$$f(t) = \sum_i A_i \cos(\omega_i t - \phi_i).$$

In order for $f(t)$ to have period 2π, each of the frequencies ω_i must be an *integer*, so

$$f(t) = \sum_{n=0}^{\infty} A_n \cos(nt - \phi_n).$$

It is more traditional to eliminate the phase shifts and just write

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

An infinite sum of this form is called a *Fourier series.*

It turns out that *any*[1] periodic function can be written as a Fourier series - this is a mathematical theorem, which we will not prove (we will come closer to proving it in math 186).

A simpler form of the Fourier expansion can be derived by expressing $\cos(x)$ and $\sin(x)$ in terms of complex exponentials, using the formulas

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}\ , \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

When we do this, we get the *complex form* of the Fourier series:

$$\begin{aligned} f(t) &= c_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{int} + e^{-int}}{2}\right) + b_n \left(\frac{e^{int} - e^{-int}}{2i}\right) \\ &= c_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{int} + \left(\frac{a_n + ib_n}{2}\right) e^{-int} \\ &= c_0 + \sum_{n=1}^{\infty} c_n e^{int} + c_{-n} e^{-int} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{int} \end{aligned}$$

Given a complex Fourier series of a real-valued function, we can always recover the real form of the Fourier series using the formulas

$$a_n = 2\mathrm{Re}[c_n]\ , \quad b_n = -2\mathrm{Im}[c_n]$$

which follow from the computations above.

In some simple cases, Fourier series can be found using purely algebraic methods. For example, to determine the Fourier series of $\cos^3(t)$, we can write

$$\cos^3(t) = \left(\frac{e^{it} + e^{-it}}{2}\right)^3 = \frac{e^{-3it} + 3e^{it} + 3e^{-it} + e^{-3it}}{8}$$

This is the complex form of the Fourier series. To find the real form, we can combine the complex exponentials $e^{\pm it}$ and $e^{\pm 3it}$:

$$\cos^3(t) = \frac{1}{4}\left(\frac{e^{-3it} + e^{3it}}{2}\right) + \frac{3}{4}\left(\frac{e^{-it} + e^{it}}{2}\right) = \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t)$$

[1]Subject to certain conditions, ask your instructor for more details and also see page 10.

If you believe that any periodic function has a Fourier series,[2] then it is surprisingly straightforward to determine the coefficients c_i. For example, to determine c_0, we can take the identity

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

and integrate both sides with respect to t from $-\pi$ to π (or any other interval of length 2π):

$$\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{int} dt = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{int} dt$$

For $n \neq 0$, we get (since $\cos(2\pi n) = 1$ and $\sin(2\pi n) = 0$ for any integer n):

$$\int_{-\pi}^{\pi} e^{int} dt = \left. \frac{e^{int}}{in} \right|_{-\pi}^{\pi} = \frac{e^{\pi in} - e^{-\pi in}}{in} = \frac{(-1)^n - (-1)^n}{in} = 0$$

So all terms of the sum with $n \neq 0$ vanish. For $n = 0$, we get

$$\int_{-\pi}^{\pi} e^{i0t} dt = \int_{0}^{2\pi} 1dt = 2\pi.$$

Substituting, we have

$$\int_{-\pi}^{\pi} f(t)dt = 2\pi c_0,$$

or

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

A nice way to think about this is that c_0 is the *average value* of $f(t)$ on the interval $[0, 2\pi]$.

A similar argument can be used to derive the general formula

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} dt$$

which allows us to calculate any coefficient in the series.[3] The integral on the right is called a *Fourier integral*.

Specifically, to derive the formula above we first write

$$f(t)e^{-int} = \sum_{k=-\infty}^{\infty} c_k e^{i(k-n)t}.$$

Then we integrate both sides, obtaining

$$\int_{-\pi}^{\pi} f(t)e^{-int} dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{i(k-n)t} dt$$

The integrals inside the sum are all zero except for the one with $n = k$, which is equal to 2π. Therefore,

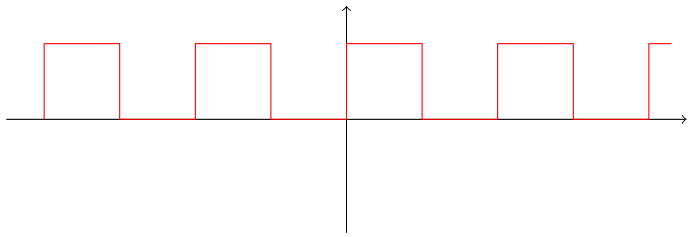
$$\int_{-\pi}^{\pi} f(t)e^{-int} dt = 2\pi c_n.$$

In math 186 you will learn about a more conceptual way of thinking about this formula for the coefficients, in terms of *orthogonal functions* and *Hilbert spaces*. For now, just think of it as a convenient observation.

[2] At one point in time, most mathematicians didn't believe it!

[3] The way that WaveEdit works is that it computes the coefficients c_n by evaluating the Fourier integrals *numerically* for a large number of values of n. It then allows you to adjust the amplitudes $|c_n|$ - this changes the shape of the graph of the function and therefore what it "sounds like".

As a simple example which can be computed explicitly, consider the *square wave* $sq(t)$, which is 2π-periodic function whose graph looks like this:



Explicitly, $sq(t)$ is given on the interval $-\pi < t < \pi$ by the formula

$$sq(t) = \begin{cases} \pi & 0 < t < \pi \\ 0 & -\pi \leq t < 0 \end{cases}$$

and for values of t outside this interval it is extended periodically (as shown above).

To calculate the Fourier coefficients of $sq(t)$, first observe that the coefficient c_0 is the average value,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} sq(t)dt = \frac{1}{2\pi}\cdot\pi^2 = \frac{\pi}{2}$$

The remaining coefficients c_n can be calculated as follows:

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} sq(t)e^{-int}dt \\ &= \int_{-\pi}^{0} 0\cdot e^{-int}dt + \frac{1}{2\pi}\int_{0}^{\pi} \pi\cdot e^{-int}dt \\ &= \frac{1}{2\pi}\int_{0}^{\pi} \pi\cdot e^{-int}dt. \\ &= \left.\frac{e^{-int}}{-2in}\right|_{0}^{\pi} \\ &= \frac{1-e^{-\pi in}}{2in} \end{aligned}$$

Since $e^{-\pi in} = -1$ for odd values of n, and $e^{-\pi in} = 1$ for even values of n.

$$c_n = \begin{cases} \frac{1}{in} & n \text{ odd} \\ 0 & n \text{ even}, n \neq 0 \\ \frac{1}{2} & n = 0 \end{cases}$$

Therefore, the Fourier series of $sq(t)$ is given by

$$\begin{aligned} sq(t) &= \cdots - \frac{e^{-5it}}{5i} - \frac{e^{-3it}}{3i} - \frac{e^{-it}}{i} + \frac{1}{2} + \frac{e^{it}}{i} + \frac{e^{3it}}{3i} + \frac{e^{5it}}{5i} + \cdots \\ &= \frac{\pi}{2} + \frac{2}{1}\sin(t) + \frac{2}{3}\sin(3t) + \frac{2}{5}\sin(5t) + \cdots \end{aligned}$$

where the real form can either be obtained using the identity

$$e^{ix} - e^{-ix} = 2i\sin(x)$$

for the values $x = t, 3t, 5t, \ldots$, or by using the relationship between real and complex Fourier coefficients,

$$b_n = -2\text{Im}[c_n].$$

Finally, we close with a simple but subtle remark. Suppose we have two Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{int}\ ,\quad \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{int}$$

which converge to the *same* function $f(t)$:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{int}$$

Then we must have

$$c_n = \tilde{c}_n.$$

If this isn't immediately obvious to you, go back to the point where we derived the formula

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

and think carefully about the argument we made - it applies equally well to *either* Fourier series above. Therefore,

$$\tilde{c}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt = c_n$$

The Frequency Domain. You would be forgiven for thinking that the formula for $sq(t)$ which we derived above is an unacceptably complicated way of expressing the numbers 0 and π. It might seem easier to describe $sq(t)$ by directly giving its values, rather than as a complicated sum of trigonometric functions.

But this would be missing the point - using Fourier series actually gives us a completely different way of thinking about functions. Rather than thinking of a function $f(t)$ as a thing which has a *value* at any given point in *time*, we can think of it as a thing which is composed of a number of oscillations, and has a different *amplitude* for every possible *frequency*. Another way to say this is that Fourier series allow us to think in the *frequency domain*, instead of the *time domain*.

To get comfortable working in the frequency domain, it helps to introduce some notation for the Fourier coefficients. For a function $f(t)$, we can write

$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

for the n^{th} Fourier coefficient. With this notation, we have

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{int}.$$

We can then ask the question, how do operations on functions which we are familiar with (sums and products, differentiation and integration, etc.) affect the Fourier coefficients?

For example, suppose we multiply $f(t)$ by a number k:

$$k \cdot f(t) = k \cdot \sum_{n=-\infty}^{\infty} \hat{f}_n e^{int} = \sum_{n=-\infty}^{\infty} k\hat{f}_n e^{int}$$

The effect of this operation is to multiply each Fourier coefficient by k.

Similarly, suppose we have two functions,

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{int} \text{ and } g(t) = \sum_{n=-\infty}^{\infty} \hat{g}_n e^{int}.$$

If we add these functions, we get

$$f(t) + g(t) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{int} + \sum_{n=-\infty}^{\infty} \hat{g}_n e^{int} = \sum_{n=-\infty}^{\infty} \left(\hat{f}_n + \hat{g}_n\right) e^{int}$$

So, when we add two functions together, we add all of their Fourier coefficients individually.

Differentiation is equally easy:

$$\frac{d}{dt}\sum_{n=-\infty}^{\infty} \hat{f}_n e^{int} = \sum_{n=-\infty}^{\infty} \frac{d}{dt}\left[\hat{f}_n e^{int}\right] = \sum_{n=-\infty}^{\infty} in\hat{f}_n e^{int}$$

$$\frac{d^2}{dt^2}\sum_{n=-\infty}^{\infty} in\hat{f}_n e^{int} = \sum_{n=-\infty}^{\infty} \frac{d}{dt}\left[in\hat{f}_n e^{int}\right] = \sum_{n=-\infty}^{\infty} (in)^2\hat{f}_n e^{int}$$

In fact, this is vastly easier than differentiating functions whose values are given to us - instead of performing all sorts of complicated rules (product rule, chain rule, etc.), all we have to do is multiply each Fourier coefficient by a simple factor.

Unfortunately, not all things are easy in the frequency domain. If we multiply two functions, here is what happens to the Fourier coefficients:

$$\begin{aligned}
f(t)g(t) &= \sum_{n=-\infty}^{\infty} \hat{f}_k e^{ikt} \sum_{l=-\infty}^{\infty} \hat{g}_l e^{ilt} \\
&= \sum_{k,l=-\infty}^{\infty} \hat{f}_k \hat{g}_l e^{ikt} e^{ilt} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k+l=n} \hat{f}_k \hat{g}_l e^{i(k+l)t} \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} \hat{f}_k \hat{g}_{n-k} \right) e^{int}
\end{aligned}$$

So, the Fourier coefficients of the product are given by an incredibly convoluted formula - in fact, this type of operation is actually called a *convolution*. We will see other examples of convolutions later.

Knowing only the rules above allows us to easily solve second order linear equations

$$my'' + by' + ky = f(t),$$

where $f(t)$ is an arbitrary periodic function. Namely, we can recast any such equation as a relationship between the Fourier coefficients $\hat{y}_n$ and $\hat{f}_n$:

$$m(in)^2 \hat{y}_n + b(in)\hat{y}_n + k\hat{y}_n = \hat{f}_n$$

Rather than solving a *differential* equation, we now must solve an *algebraic* equation, which is easy:

$$\hat{y}_n = \frac{\hat{f}_n}{k - mn^2 + inb}$$

Now that we have the Fourier coefficients of $y(t)$, we can convert back to the time domain:

$$y(t) = \sum_{n=-\infty}^{\infty} \hat{y}_n e^{int} = \sum_{n=-\infty}^{\infty} \frac{\hat{f}_n e^{int}}{k - mn^2 + inb}$$

As long as b and k are positive, the denominators in this formula will be nonzero, and the sum will converge to a solution of the equation we wanted to solve!

Now, you might want to actually evaluate the sum on the right hand side - good luck to you in that case! Often it is better to use a computer to sum the first 100 terms and have it graph the result. Again, the point of using Fourier series is not to understand the values of the function, but rather to understand the oscillations which that function is composed of.

Sine and Cosine Series. You are probably familiar with the concept of *even* and *odd* functions. An even function is a function which satisfies the identity

$$f(-t) = f(t)$$

and an odd function is one which satisfies the opposite identity,

$$f(-t) = -f(t)$$

For example, any even power $t^2, t^4, t^6, \dots$ is an even function, and any odd power $t, t^3, t^5, \dots$ is odd.

Likewise, the function $\sin(t)$ is odd, whereas $\cos(t)$ is even.

The function e^t is *neither* even nor odd, but it *is* the sum of an even function and an odd function:

$$e^t = \frac{e^t + e^{-t}}{2} + \frac{e^t - e^{-t}}{2} = \cosh t + \sinh(t)$$

It's a fun exercise to prove that *any* function can be written in exactly one way as the sum of an even function and an odd function (called the *even part* and the *odd part* of the function).

In the case of a periodic function, we can write

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

If we replace t with $-t$, we see that

$$f(-t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) - b_n \sin(nt).$$

Therefore, in order for $f(t)$ to be *even*, we must have

$$c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) - b_n \sin(nt)$$

and therefore $b_n = -b_n$ for every n, i.e. $b_n = 0$.

For $f(t)$ to be *odd*, we must have $a_n = c_0 = 0$ for every n, by similar reasoning.

It follows that any even periodic function $g(t)$ can be represented as a *cosine series*,

$$g(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt),$$

where the coefficients a_n are given by

$$a_n = 2\text{Re}\,[c_n] = \text{Re}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\right] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

Similarly, any odd periodic function $h(t)$ can be represented as a *sine series*,

$$h(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

where the coefficients b_n are given by

$$b_n = -2\text{Im}\,[c_n] = -\text{Im}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} h(t)e^{-int}\right] = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\sin(nt)dt.$$

In either case, half of the coefficients vanish, so computing them using the real formula is the same amount of work as computing the complex coefficients using the complex formula.

There is a further trick which makes the task of computing the coefficients even simpler. Supppose that $g(t)$ is an even function. Then the functions $g(t)\cos(nt)$ are all even functions (exercise: the product of any two even functions is even!) So, to compute the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)dt$$

we actually only have to compute the integral from 0 to π and multiply the result by 2:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)dt = \frac{2}{\pi}\int_{0}^{\pi} g(t)\cos(nt)dt.$$

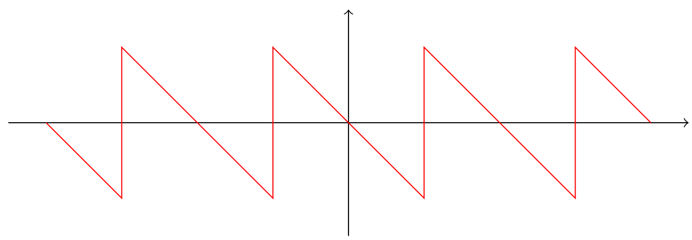
Similarly for an odd function $h(t)$, the functions $h(t)\sin(nt)$ are all even (exercise: the product of two odd functions is always an even function!). So in this case,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\sin(nt)dt = \frac{2}{\pi}\int_{0}^{\pi} h(t)\sin(nt)dt.$$

For example, consider the *sawtooth wave*, which is given by the formula

$$sw(t) = -t$$

on the interval $[-\pi, \pi]$, and then extended to have period 2π. Drawing the graph, we see that $sw(t)$ is odd:



So, to compute its Fourier series we need only compute the coefficients b_n (all a_n are zero automatically):

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} -t\sin(nt)dt \\ &= \frac{2}{\pi}\left(\frac{t\cos(nt)}{n}\right)\Big|_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\cos(nt)}{n}dt \\ &= \frac{2}{\pi}\cdot\frac{\pi\cos(n\pi)}{n} - 0 \\ &= 2\cdot\frac{(-1)^n}{n} \end{aligned}$$

This gives us the Fourier series,

$$sw(t) = 2\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nt)$$

Notice that something strange happens if we substitute $t = \pi$ in this formula:

$$2\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi) = 2\cdot\sum_{n=1}^{\infty} 0 = 0$$

This is different from the value of $sw(t)$ at $t = 0$! More precisely, $sw(t)$ has two different limiting values as $t \to \pi$, but neither of them is 0.

What is going on here is that the Fourier series is *averaging* the two different limits. One of the limits is -1 and the other is $+1$,

$$\lim_{t\to\pi^+} sw(t) = 1\ ,\quad \lim_{t\to\pi^-} sw(t) = -1.$$

so the Fourier series splits the difference and gives us the value 0.

As long as $-\pi < t < \pi$, the Fourier series actually gives the correct value. For example,

$$\sum_{n=1}^{\infty}(-1)^n\frac{2}{n}\sin\left(\frac{n\cdot\pi}{2}\right) = -\frac{2}{1}+\frac{2}{3}-\frac{2}{5}+\frac{2}{7}-\cdots = -2\tan^{-1}(1) = -2\cdot\frac{\pi}{4} = -\frac{\pi}{2} = sw\left(\frac{\pi}{2}\right).$$

Here we have evaluated the infinite sum using the Taylor series of $\tan^{-1}(x)$:

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

In general, it is only valid to evaluate Fourier series at points where the function is *differentiable* - at points where the function or its derivative are discontinuous, the series may not converge, or it may converge to the incorrect value. To learn about the details of this, you must take a course in mathematical analysis.

However, we can state the following theorem:

Convergence of Fourier Series. Let $f(t)$ be a function which is 2π- periodic and differentiable except at finitely many points in the interval $[-\pi, \pi]$. Also suppose that at every point where $f(t)$ is *not* differentiable, the limits

$$f(a^+) = \lim_{t \to a^+} f(t)\ ,\quad f(a^-) = \lim_{t \to a^-} f(t)$$

exist. Then at every point $t = t_0$ where $f(t)$ is continuous (i.e. $f(t_0^+) = f(t_0^-)$), the Fourier series of $f(t)$ converges to the value $f(t_0)$:

$$f(t_0) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt_0) + b_n \sin(nt_0)$$

and at any other point, it converges to the average of the upper and lower limiting values,

$$\frac{f(t_0^+) + f(t_0^-)}{2} = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt_0) + b_n \sin(nt_0)$$

In this theorem it is crucial that the function $f(t)$ is 2π- periodic. For a function which is not periodic, the identity

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

is only guaranteed to be valid for values of t in the range

$$-\pi < t < \pi.$$

Fortunately, there is a trick for turning arbitrary functions into periodic ones! Actually, there are two tricks, which get used in different contexts.

Given *any* function $h(t)$ on the interval $[0, \pi]$, we can first extend it to an odd function on $[-\pi, \pi]$ by defining

$$h_{odd}(t) = \begin{cases} h(t) & 0 < t < \pi \\ -h(-t) & -\pi < t < 0 \end{cases}$$

on the interval $[-\pi, \pi]$, and then we can extend it to a periodic function by defining

$$h_{odd}(t + 2\pi n) = h_{odd}(t)$$

for arbitrary multiples of 2π. The result is an odd, 2π-periodic function $h_{odd}(t)$, which is equal to $h(t)$ for all values of t in the interval $[0, \pi]$.

Similarly, we can extend $h(t)$ to an *even* function by defining

$$h_{even}(t) = \begin{cases} h(t) & 0 < t < \pi \\ h(-t) & -\pi < t < 0 \end{cases}$$

on the interval $[-\pi, \pi]$, and then we can extend it to a periodic function by defining

$$h_{even}(t + 2\pi n) = h_{even}(t)$$

for arbitrary multiples of 2π. The result is an even, 2π-periodic function $h_{even}(t)$, which is equal to $h(t)$ on the interval $[0, \pi]$.

It follows that the formulas

$$h(t) = \sum_{n=1}^{\infty} b_n \sin(nt)\ ,\quad b_n = \frac{2}{\pi} \int_0^{\pi} h(t) \sin(nt) dt$$

$$h(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)\ ,\quad a_n = \frac{2}{\pi} \int_0^{\pi} h(t) \cos(nt) dt$$

are valid for *arbitrary differentiable functions* on the interval $[0,\pi]$, except possibly at the endpoints 0 and π, where the odd/even periodic extension may have introduced discontinuities in the function.

There is one case when the odd periodic extension has a continuous derivative (and so the sine series of $h(t)$ gives the correct value, even at the endpoints). Namely, if $h(0) = h(\pi) = 0$, then when we make the odd periodic extension, the values of $h(0)$ and $h'(0)$ match up on both sides (and similarly at $t = \pi$).

Similarly, if $h'(0) = h'(\pi) = 0$, then the even periodic extension has a continuous derivative (and so the cosine series of $h(t)$ gives the correct value, even at the endpoints.

Odd periodic extensions are therefore useful for solving *boundary value problems* of the form

$$my'' + ky = f(t), \quad y(0) = y(\pi) = 0,$$

where $f(t)$ is a function on $[0,\pi]$. Boundary value problems of this type are called *Dirichlet* problems.

Simiarly, even periodic extensions are useful for solving boundary value problems of the form

$$my'' + ky = f(t), \quad y'(0) = y'(\pi) = 0.$$

Boundary value problems of this type are called *Neumann* problems.

To solve a Dirichlet problem, we can first expand $f(t)$ as a sine series,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt).$$

We then solve each of the equations

$$my_n'' + ky_n = b_n \sin(nt)$$

individually, and see that the solution takes the form

$$y_n = B_n \sin(nt)$$

for some value of B_n. This gives a particular solution,

$$y = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} B_n \sin(nt),$$

whose values at $t = 0$ and $t = \pi$ are both clearly 0!

To solve a Neumann problem, we can first expand $f(t)$ as a cosine series,

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt)$$

and then solve the equations

$$my_0'' + ky_0 = c_0$$
$$my_n'' + ky_n = a_n \cos(nt)$$

The solutions take the form

$$y_0 = C_0$$
$$y_n = A_n \cos(nt)$$

and we obtain a particular solution

$$y(t) = \sum_{n=0}^{\infty} y_n = C_0 + \sum_{n=1}^{\infty} A_n \cos(nt)$$

whose derivative

$$y'(t) = \sum_{n=1}^{\infty} -nA_n \sin(nt)$$

satisfies the boundary condition $y'(0) = y'(\pi) = 0$.