Math 18500 Week 4: Second Order Homogeneous Equations

Linear Differential Operators. We have seen that it can be useful to distinguish between two types of differential equations - *linear* and *nonlinear*. First order linear equations and first order systems of linear equations are much easier to solve than nonlinear ones, and their solutions have nicer properties. The same is true of higher order linear equations as well.

A differential equation of order n is said to be *linear* if it can be written in the form

$$p_n(t)\frac{d^ny}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t)$$

where $p_0(t), \ldots, p_n(t)$ and $f(t)$ are arbitrary functions of t. Any other equation is said to be *nonlinear*.

There is a very rich theory of linear equations, which incorporates a number of robust and useful concepts. By contrast, the theory of nonlinear equations is mostly restricted to a few special examples which can only be solved approximately, or by a combination of ingenuity and physical/geometric insight. For this reason, we will focus on linear equations for the remainder of the course.

When thinking about linear equations, it is useful to introduce the language of *linear differential operators*.

The simplest linear differential operator is the derivative operator, $\frac{d}{dt}$. It is an *operator* because it *operates* on functions: given a function $y(t)$, we can apply the operator $\frac{d}{dt}$, and the result is a new function,

$$\frac{d}{dt}\left[y(t)\right] = y'(t).$$

This operation is *linear*, because it obeys the sum rule:

$$\frac{d}{dt}\left[y_1(t) + y_2(t)\right] = \frac{d}{dt}\left[y_1(t)\right] + \frac{d}{dt}\left[y_2(t)\right]$$

and more generally,

$$\frac{d}{dt}\left[c_1y_1(t) + c_2y_2(t)\right] = c_1\frac{d}{dt}\left[y_1(t)\right] + c_2\frac{d}{dt}\left[y_2(t)\right]$$

for any functions y_1 and y_2 and constants c_1 and c_2.

When we apply the derivative operator *twice*, the result is a second derivative:

$$\frac{d}{dt}\left[\frac{d}{dt}\left[y\right]\right] = \frac{d^2y}{dt^2} = y''$$

For second and higher derivatives it is useful to introduce the shorthand

$$D = \frac{d}{dt},$$

because the notation

$$D^2y = y'' \ , \quad D^3y = y''' \ , \quad \text{etc.}$$

is a bit easier on the eyes than

$$\frac{d^2y}{dt^2} \ , \quad \frac{d^3y}{dt^3} \quad \text{etc.},$$

and it more accurately reflects the idea that we are differentiating several times in a row.

In general, a linear differential operator is any operator of the form

$$\mathcal{O} = p_nD^n + p_{n-1}D^{n-1} + \cdots + p_2D^2 + p_1D + p_0,$$

where $p_i = p_i(t)$ can be arbitrary functions of t.

When we apply an operator like this to a function $y(t)$, we get a new function,

$$\begin{aligned}\mathcal{O}y &= p_nD^ny + p_{n-1}D^{n-1}y + \cdots + p_2D^2y + p_1Dy + p_0y \\ &= p_n\frac{d^ny}{dt^n} + p_{n-1}\frac{d^{n-1}y}{dt^{n-1}} \cdots + p_2\frac{d^2y}{dt^2} + p_1\frac{dy}{dt} + p_0y\end{aligned}$$

and this operation is linear:

$$\mathcal{O}\left[c_1y_1(t) + c_2y_2(t)\right] = c_1\mathcal{O}\left[y_1(t)\right] + c_2\mathcal{O}\left[y_2(t)\right]$$

With this notation, any linear differential equation can be written conceptually in the form

$$\mathcal{O}y = f,$$

where $\mathcal{O}$ is a linear differential operator. This is helpful for the same reason that it is helpful to write systems of linear equations in the form

$$A\vec{x} = \vec{b}$$

where A is a matrix - the abstraction helps us organize our thoughts and makes it clearer what's going on.

For example, operator notation allows to clearly identify three properties of linear equations:

(1) Suppose that $y_1(t)$ and $y_2(t)$ are two solutions of a linear homogeneous equation:

$$\mathcal{O}y_1 = 0$$
$$\mathcal{O}y_2 = 0.$$

If c_1 and c_2 are arbitrary constants, and

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

then $y(t)$ is a solution of the same homogeneous equation:

$$\mathcal{O}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{O}y_1 + c_2 \mathcal{O}y_2 = 0 + 0 = 0$$

In other words, *solutions of homogeneous equations can be superimposed.*

(2) Suppose that $y_p(t)$ is a *particular* solution of an inhomogeneous equation,

$$\mathcal{O}y_p = f$$

If $y_h(t)$ is any solution of the corresponding *homogeneous* equation,

$$\mathcal{O}y_h = 0.$$

and if

$$y(t) = y_p(t) + y_h(t),$$

then $y(t)$ is a solution of the original inhomogeneous equation:

$$\mathcal{O}y = \mathcal{O}y_p + \mathcal{O}y_h = f + 0 = f$$

(3) Suppose that $y_1(t)$ and $y_2(t)$ are solutions of linear inhomogeneous equations with different right hand sides:

$$\mathcal{O}y_1 = f_1$$
$$\mathcal{O}y_2 = f_2$$

If c_1 and c_2 are arbitrary constants, and

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

then $y(t)$ is a solution of the inhomogeneous equation

$$\mathcal{O}\left[c_1 y_1 + c_2 y_2\right] = c_1 f_1 + c_2 f_2$$

where the right hand side is a linear combination of the right hand sides of the original equations.

We will see that these properties are crucial for solving linear equations.

Second Order Homogeneous Equations: Exponential Ansatz Method. In the next two sections we will be focused on solving second order homogeneous equations

$$y'' + py' + qy = 0$$

whose coefficients p and q are *constants*. An equation like this can also be written using operator notation,

$$D^2y + pDy + qy = 0,$$

or more succinctly as

$$\mathcal{O}y = 0$$

where

$$\mathcal{O} = D^2 + pD + q.$$

Most equations of this form can be solved using two key ideas:

1. The Exponential Ansatz.
2. The Superposition Principle.

You may not have seen the word *ansatz* before. It's a German word, which means something like "educated guess". Probably it would be more accurate to say "lucky guess" - at least, this is what an ansatz usually looks like to those who don't already know the answer.

For second order homogeneous equations with constant coefficients, our lucky guess will take the form

$$y(t) = e^{\lambda t},$$

where λ is an unspecified number (possibly complex). A guess of this form is called an *exponential ansatz.*

To see why the exponential ansatz is a good guess, recall that the identity

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$$

holds for any real or complex number λ. Similarly for the higher derivatives, we have

$$D^k e^{\lambda t} = \frac{d^k}{dt^k}e^{\lambda t} = \lambda^k e^{\lambda t}.$$

Therefore, if we make the substitution $y = e^{\lambda t}$ in the equation

$$D^2y + pDy + qy = 0$$

we get

$$\lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = 0.$$

Dividing by $e^{\lambda t}$, we obtain a quadratic equation for λ:

$$\lambda^2 + p\lambda + q = 0$$

The polynomial on the left hand side is called the *auxiliary polynomial* of the differential operator

$$\mathcal{O} = D^2 + pD + q.$$

In most cases, the auxiliary polynomial will have 2 distinct roots, λ_1 and λ_2 (which may be real or complex). Therefore, we usually find two *exponential solutions*

$$y_1(t) = e^{\lambda_1 t} \text{ and } y_2(t) = e^{\lambda_2 t}.$$

when we make an exponential ansatz.

For example, consider the equation

$$y'' + 4y' + 3y = 0$$

In this case, the auxiliary equation is

$$\lambda^2 + 4\lambda + 3 = 0,$$

which factors as

$$(\lambda + 1)(\lambda + 3) = 0,$$

and therefore has solutions $\lambda_1 = -1$ and $\lambda_2 = -3$. The corresponding exponential solutions are

$$y_1(t) = e^{-t} \text{ and } y_2(t) = e^{-3t}.$$

As another example, consider the equation

$$y'' + 4y' + 5y = 0.$$

In this case, the auxiliary equation is

$$\lambda^2 + 4\lambda + 5 = 0,$$

which has two *complex roots*:

$$\lambda = \frac{-2 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = -2 \pm i.$$

In this case, the exponential solutions are *complex-valued* functions,

$$y(t) = e^{(-2+i)t} = e^{-2t}e^{it} = e^{-2t}(\cos t + i \sin t) = e^{-2t}\cos t + ie^{-2t}\sin t$$

and

$$\overline{y}(t) = e^{(-2-i)t} = e^{-2t}e^{-it} = e^{-2t}(\cos t - i \sin t) = e^{-2t}\cos t - ie^{-2t}\sin t.$$

However, in most applications we want to have *real* solutions - the complex solutions are not so useful.

To obtain two real solutions, recall that any complex equation can be split into real and imaginary parts. In the example above, writing the complex solution as $y = y_1 + iy_2$, the equation that it satisfies is

$$(D^2 + 4D + 5)(y_1 + iy_2) = 0.$$

Applying linearity, we find that

$$(D^2 + 4D + 5)y_1 + i(D^2 + 4D + 5)y_2 = 0 + 0i.$$

Splitting this equation into its real and imaginary parts, we conclude that

$$(D^2 + 4D + 5)y_1 = 0$$

and

$$(D^2 + 4D + 5)y_2 = 0$$

Therefore, the real and imaginary parts of the complex solution,

$$y_1(t) = e^{-2t}\cos t \;\text{ and }\; y_2(t) = e^{-2t}\sin t,$$

are each *individually* solutions of the original differential equation.

Alternatively, this follows from the superposition principle, since y_1 and y_2 are both (complex) linear combinations of y and $\overline{y}$:

$$y_1(t) = \frac{y(t) + \overline{y}(t)}{2} \;\text{ and }\; y_2(t) = \frac{y(t) - \overline{y}(t)}{2i}.$$

and solutions of homogeneous equations can always be superimposed.

In either case (real or complex), we end up with two real solutions $y_1(t)$ and $y_2(t)$.[1]. Since $y_1(t)$ and $y_2(t)$ are both solutions of the linear homogeneous equation

$$\mathcal{O}[y(t)] = 0,$$

the superposition principle tells us that any linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$ is a solution of the same homogeneous equation:

$$\mathcal{O}[c_1y_1 + c_2y_2] = c_1\mathcal{O}[y_1] + c_2\mathcal{O}[y_2] = 0 + 0 = 0.$$

We can use this to construct solutions with arbitrary initial values $y(t_0)$ and $y'(t_0)$ at any initial time t_0.

For example, let's use the superposition principle to solve the initial value problem

$$y'' + 4y' + 3y = 0 \;,\; y(0) = 0 \;,\; y'(0) = 2.$$

We have seen that the exponential solutions of this equation are

$$y_1(t) = e^{-t} \;\text{ and }\; y_2(t) = e^{-3t}.$$

Therefore, any linear combination

$$y(t) = c_1e^{-t} + c_2e^{-3t}$$

[1] There is an exception when the characteristic equation has a *repeated root* - we will come back to this case later.

is also a solution. The derivative of this solution is

$$y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}.$$

To achieve the correct initial conditions, we set $t = 0$, obtaining a system of equations for c_1 and c_2:

$$y(0) = c_1 + c_2 = 0$$
$$y'(0) = -c_1 - 3c_2 = 2.$$

Adding the two equations, we find that

$$c_1 - c_1 + c_2 - 3c_2 = 2,$$

so $c_2 = -1$. Multiplying the first equation by 3 and adding it to the second, we see that

$$3c_1 - c_1 + 3c_2 - 3c_2 = 2,$$

so $c_1 = 1$. Therefore, the solution with the correct initial values is

$$y(t) = e^{-t} - e^{-3t}.$$

Similarly, we can solve an initial value problem like

$$y'' + 4y' + 5y = 0\ ,\ y(0) = 1\ ,\ y'(0) = -1$$

by taking y to be a linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t,$$

and proceeding as above (differentiate y, set $t = 0$, solve the system of equations for c_1 and c_2...).

Second Order Homogeneous Equations: Repeated Integration Method (reading). There is one special case where the exponential ansatz method fails to produce a general solution of a given second order equation: when the auxiliary equation has a repeated root.

For example, consider the equation

$$y'' + 4y' + 4y = 0$$

In this case the auxiliary equation is

$$\lambda^2 + 4\lambda + 4 = 0,$$

which factors as

$$(\lambda + 2)^2 = 0,$$

so the equation has a repeated root,

$$\lambda = -2.$$

As a result, the exponential ansatz method produces only one solution,

$$y_1 = e^{-2t}.$$

To find a second solution of the equation, we need a new method.

The idea behind the second method is to rewrite the equation using *operator notation*:

$$(D^2 + 4D + 4)y = 0.$$

We can then *factor the operator* on the left hand side:

$$D^2 + 4D + 4 = (D + 2)^2.$$

Now, you should immediately be suspicious here, because we have seen in 183 that the ordinary algebra we're used to doesn't necessarily apply to operators. For example, matrices can be thought of as operators on vectors, and we know that matrix multiplication is not commutative:

$$AB \neq BA$$

One consequence of this is that the usual binomial formula is invalid for matrices (and other kinds of operators as well). Indeed,

$$(A + B)^2 = (A + B)(A + B) = A(A + B) + B(A + B) = A^2 + AB + BA + B^2$$

and this is not equal to

$$A^2 + 2AB + B^2$$

unless the two matrices commute (i.e. $AB = BA$).

Fortunately, a similar issue does not appear here. Indeed,

$$(2D)y = 2(Dy) = 2y' = (2y)' = D(2y) = (D2)y,$$

so the operators $2D$ and $D2$ are equal, and we can conclude that

$$(D + 2)^2 = D^2 + 2D + D2 + 2^2 = D^2 + 4D + 4$$

Note that this would be invalid for second order operators with *nonconstant* coefficients! For example,

$$(D^2 + 2tD + t^2)y = y'' + 2ty' + t^2y$$

but on the other hand

$$(D + t)(D + t)y = (D + t)(y' + ty) = y'' + (ty)' + ty' + t^2y = y'' + y + ty' + ty' + t^2y = y'' + 2ty' + (1 + t^2)y.$$

So we are using something very special about constant coefficient operators, when we factor like this.

Returning to the problem at hand, we now see that it is valid to replace the equation

$$(D^2 + 4D + 4)y = 0$$

with the equation

$$(D + 2)(D + 2)y = 0.$$

This is helpful, because if we make the substitution

$$(D + 2)y = u$$

then we are left with a *first order* equation,

$$(D+2)u = 0$$

which we know how to solve!

Namely, if we convert back from operator notation we get

$$\frac{du}{dt} + 2u = 0$$

and this is a separable equation, whose solution we know:[2]

$$u = c_2 e^{-2t}.$$

Substituting into the equation

$$(D+2)y = u,$$

we obtain

$$y' + 2y = c_1 e^{-2t}$$

and this is a first order linear equation, which can be solved using integrating factors!

Multiplying by the integrating factor $J = e^{2t}$, and proceeding with the standard method, we obtain

$$e^{2t}y' + 2e^{2t}y = e^{2t}e^{-2t}c_1 = c_1$$

$$\frac{d}{dt}\left[e^{2t}y\right] = c_1$$

$$e^{2t}y = c_1 t + c_2$$

$$y = (c_1 t + c_2)e^{-2t} = c_1 t e^{-2t} + c_2 e^{-2t}$$

If you work this out in complete generality, you will find that in the case of a repeated root, the equation

$$(D+\lambda)(D+\lambda)y = 0$$

has the general solution

$$y = c_1 t e^{\lambda t} + c_2 e^{\lambda t}.$$

The same method above can be applied to *arbitrary* second order equations with constant coefficients - this method is referred to as *repeated integration*. In general, to solve an equation

$$(D^2 + pD + q)y = 0,$$

we factor the right hand side,

$$(D-\lambda_1)(D-\lambda_2)y = 0$$

and then make the substitution

$$(D-\lambda_2)y = u.$$

This leaves us with the equation

$$(D-\lambda_1)u = 0.$$

We can then solve for u and v using our standard techniques for solving first order equations (separating the variables, and integrating factors). If you carry this out you will find that the solutions are

$$u = c_1 e^{\lambda_1 t}$$

and (if the roots are distinct),

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

This derivation confirms the validity of the exponential ansatz method - it always produces the most general solution possible!

[2] You'll see why we're calling the constant of integration c_2 in a second.

Damped Oscillators. We have seen how to solve linear equations with constant coefficients, but the solution was rather unmotivated - we have not explained how such an equation would come up in practice. So, it is worth introducing some physical examples which you can have in mind when thinking about second order linear equations.

For the first example, consider a mass hanging on a spring. The mass has an equilibrium position, in which it is at rest - let y denote the vertical displacement from this equilibrium position. Then combining Newton's law for the force acting on an object of mass m,

$$F = ma = my''$$

with Hooke's law for the force exerted by a spring,

$$F = -ky$$

we see that $y(t)$ satisfies the equation

$$m\frac{d^2y}{dt^2} + ky = 0.$$

This homogeneous and we know how to solve it - the corresponding characteristic equation

$$m\lambda^2 + k = 0$$

has purely imaginary roots

$$\lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0,$$

so a fundamental pair of solutions is

$$y_1(t) = \text{Re}\left[e^{i\omega_0 t}\right] = \cos(\omega_0 t) \text{ and } y_2(t) = \text{Im}\left[e^{i\omega_0 t}\right] = \sin(\omega_0 t),$$

and the general solution is

$$y(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t).$$

The constant ω_0 is called the *natural frequency* of the mass-spring system.

Sometimes it is nice to write the solution $y(t)$ in *phase-shifted* form

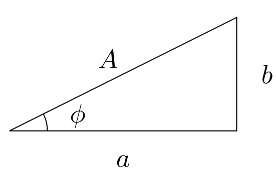
$$y(t) = A\cos(\omega_0 t - \phi).$$

In this context, A is called the *amplitude* of the oscillation, and ϕ is called the *phase shift.*

To bring the solution into phase-shifted form, we first make the substitution

(1) $$a = A\cos\phi \ , \ b = A\sin\phi,$$

or equivalently we draw a triangle:



We then apply the angle subtraction formula, $\cos(x - y) = \cos x\cos y + \sin x\sin y$:

$$y(t) = A\cos(\omega_0 t)\cos(\phi) + A\sin(\omega_0 t)\sin(\phi) = A\cos(\omega_0 t - \phi)$$

Another method is to write $y(t)$ as the real part of a complex function,

$$(a - bi)e^{i\omega_0 t} = (a - bi)(\cos\omega_0 + i\sin\omega_0) = (a\cos\omega_0 t + b\sin\omega_0 t) + i(-b\cos(\omega_0 t) + a\sin(\omega_0 t)).$$

Making the same subsitution as before, we have

$$a - bi = A\cos\phi - iA\sin\phi = Ae^{-i\phi}$$

and therefore

$$(a - bi)e^{i\omega_0 t} = Ae^{-i\phi}e^{i\omega_0 t} = Ae^{i(\omega_0 t - \phi)}.$$

Equating real parts gives us

$$a\cos\omega_0 t + b\sin\omega_0 t = \text{Re}[Ae^{i(\omega_0 t - \phi)}] = A\cos(\omega_0 t - \phi).$$

Unfortunately, a sinusoidal solution is not physically realistic - the energy of an actual spring would be dissipated gradually in the form of heat, and the spring would eventually come back to rest at its equilibrium position. This effect can be modeled[3] by introducing a *damping force* which resists the *velocity* of the spring, rather than its displacement from equilibrium. Adding a damping force leads to an equation of the form

$$m\frac{d^2y}{dt^2} + l\frac{dy}{dt} + ky = 0,$$

where m, l, and k are positive constants.

In the damped case, the characteristic equation

$$m\lambda^2 + l\lambda + k = 0$$

has two roots

$$\lambda = \frac{-l \pm \sqrt{l^2 - 4mk}}{2m} = -b \pm \sqrt{b^2 - \omega_0^2},$$

where

$$b = \frac{l}{2m} \text{ and } \omega_0 = \sqrt{\frac{k}{m}}$$

Note that the roots may be either real or complex, depending on whether $b^2 - \omega_0^2$ is positive or negative.

In the real case ($b^2 > \omega_0^2$), the system is said to be *overdamped*, and the corresponding differrential equation has two real exponential solutions,

$$y_1 = e^{r_1 t} \ \text{ and } \ y_2 = e^{r_2 t}.$$

Since b and ω_0^2 are positive, we have

$$\pm\sqrt{b^2 - \omega_0^2} < \sqrt{b^2} = b,$$

which implies that

$$-b \pm \sqrt{b^2 - \omega_0^2} < -b + b = 0.$$

Therefore, the roots r_1 and r_2 are both negative, and the corresponding exponential solutions are *decreasing*.

In the complex case ($b^2 < \omega_0^2$), the system is said to be *underdamped*, and the exponential solutions are

$$y = e^{(r+i\omega)t} \quad , \quad y = e^{(r-i\omega)t}$$

where

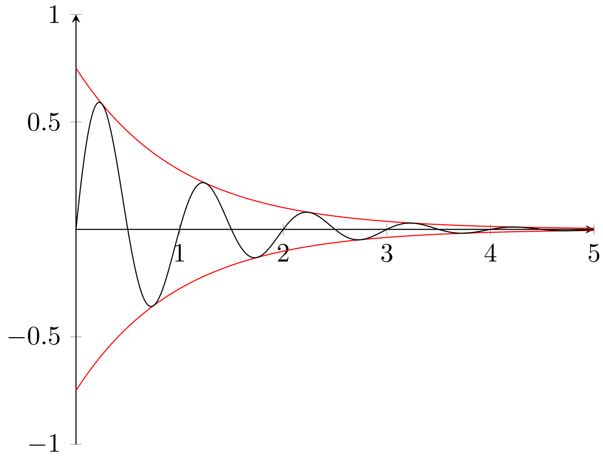
$$r = -b \ \text{ and } \ \omega = \sqrt{\omega_0^2 - b^2}.$$

The real and imaginary parts of the complex solutions are a fundamental pair of solutions,

$$y_1 = e^{rt}\cos(\omega t) \ \text{ and } \ y_2 = e^{rt}\sin(\omega t).$$

[3]In reality, the damping of a mass-spring system is nonlinear - the model we are describing is only an approximation, and this approximation is only accurate when the system is very close to its equilibrium state.

To graph these functions, one draws an "exponential envelope" $y = \pm e^{rt}$, together with a sinusoid oscillating between the upper and lower boundaries of the envelope:



Note that both solutions still decay to 0 exponentially (since $r = -b < 0$), but now they also oscillate as they decay. Also note that damping reduces the frequency of the oscillation.

There is also a third case - if $b^2 = \omega_0^2$, then the characteristic equation has a repeated root and the oscillator is said to be *critically damped*. In this case, a fundamental pair of solutions is

$$y_1 = e^{rt} \quad , \quad y_2 = te^{rt},$$

where

$$r = -b < 0.$$

Note that both of these solutions decay to zero as well.

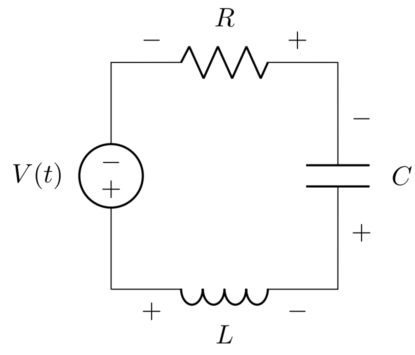
With this model in mind, we can also understand the physical meaning of an *inhomogeneous* equation

$$my'' + ly' + ky = f(t).$$

Such an equation describes a mass on a spring which is subject to an additional *external force* $f(t)$.

Any physical system which can be accurately modeled by an equation of this form is called a *damped oscillator* (in the case $l > 0$) or a *harmonic oscillator* (in the case $l = 0$).

Damped oscillators need not be *mechanical* systems. For example, consider a circuit with a resistor, capacitor, inductor, and power source:



A circuit like this can be modeled by

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V(t),$$

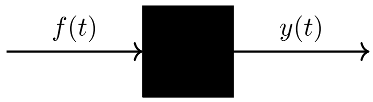
where R, L, C denote resistance, inductance, and capacitance respectively, $I(t)$ is the current flowing through the circuit, $Q(t)$ is the charge built up on the capacitor, and $V(t)$ is the voltage supplied by the power source.

Differentiating this equation gives an equation for the current,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = V'(t).$$

From a mathematical point of view, this electrical system is completely equivalent to the mechanical mass-spring system, because the two systems can be modeled by equations of exactly the same form.

We would like to apply the physical intuition provided by either of these models to our understanding of second order linear equations, but of course there is no reason prefer one model over any other. For this reason, it is common to visualize a damped oscillator using the following "black box" diagram:



In this picture, $f(t)$ represents the *input* to our physical system (e.g. external forces in the mass-spring case, or the derivative of voltage in the case of an RLC circuit), and $y(t)$ represents the *response* to that input. To find the response, we must solve an equation of the form

$$my''(t) + ly'(t) + ky(t) = f(t),$$

subject to whatever initial conditions happen to be physically reasonable.

Likewise, when we solve an inhomogeneous differential equation, we can always frame our solution as the answer to a physical-sounding question: how does the system respond to a particular given input? We will be particularly interested in this question in two special cases: first, when the input is a *sinusoid* of a particular frequency, and second, when the input is a very brief *impulse*. We will investigate both cases in detail in the coming weeks, and these investigations will lead us to some very interesting mathematical ideas!