Lecture 11: Other discrete random variables

Statistics 251

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Where are we?

Geometric random variables

Negative binomial random variables

Example

Geometric random variables

Consider independent tosses of a coin which is heads with probability p. A **geometric** random variable with parameter p is the number of tosses before the first appearance of heads.

We have that

$$\mathbb{P}(X = k) = p(1-p)^{k-1}.$$

Properties of geometric random variables

For a geometric random variable, we have

$$\mathbb{P}(X = k) = p(1-p)^{k-1}.$$

What is $\mathbb{E}[X]$?

By definition, we have

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} kp(1-p)^{k-1}.$$

Notice that

$$\begin{aligned}\mathbb{E}[X] - 1 &= \sum_{k=1}^{\infty} kp(1-p)^{k-1} - \sum_{k=1}^{\infty} p(1-p)^{k-1} \\ &= \sum_{k=1}^{\infty}(k-1)p(1-p)^{k-1} = \sum_{j=1}^{\infty} jp(1-p)^{j} = (1-p)\mathbb{E}[X].\end{aligned}$$

We can solve to find $\mathbb{E}[X] = 1/p$.

Properties of geometric random variables

For a geometric random variable, we have

$$\mathbb{P}(X = k) = p(1-p)^{k-1}.$$

What is $\mathrm{Var}(X)$?

Notice that

$$\mathbb{E}[(X-1)^2] = \sum_{k=1}^{\infty}(k-1)^2 p(1-p)^{k-1} = \sum_{j=1}^{\infty} j^2 p(1-p)^j = (1-p)\mathbb{E}[X^2].$$

We find that

$$p\mathbb{E}[X^2] = 2\mathbb{E}[X] - 1 = \frac{2}{p} - 1 \implies \mathbb{E}[X^2] = \frac{2}{p^2} - \frac{1}{p}.$$

We conclude that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1-p}{p}$.

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Negative binomial random variables

Consider independent tosses of a coin which is heads with probability p. A **negative binomial** random variable with parameters r and p is *the toss number until(and include) the* r*th head.*

To get the rth head on the kth toss, we need exactly $r-1$ heads among the first $k-1$ tosses. There are $\binom{k-1}{r-1}$ ways to choose these tosses, implying

$$\mathbb{P}(X = k) = \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r} p.$$

Negative binomial properties

Consider independent tosses of a coin which is heads with probability p. A **negative binomial** random variable with parameters r and p is the toss number of the rth head.

What is $\mathbb{E}[X]$? Notice that

$$X = X_1 + X_2 + \cdots + X_r,$$

where X_i is geometric with parameter p. This implies that

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_r] = \frac{r}{p}.$$

Negative binomial properties

Consider independent tosses of a coin which is heads with probability p. A **negative binomial** random variable with parameters r and p is the toss number of the rth head.

What is $\mathrm{Var}(X)$? Notice that $X = X_1 + X_2 + \cdots + X_r$ where X_i is geometric with parameter p. This implies that

$$\begin{aligned}\mathbb{E}[X^2] = \mathbb{E}\left[\sum_{i,j=1}^{r} X_i X_j\right] &= \sum_{i=1}^{r} \mathbb{E}[X_i^2] + 2 \sum_{1 \le i < j \le r} \mathbb{E}[X_i X_j] \\ &= r\frac{2-p}{p^2} + r(r-1)\frac{1}{p^2}.\end{aligned}$$

We find

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = r\frac{2-p}{p^2} - \frac{r}{p^2} = \frac{r(1-p)}{p^2}.$$

Alternative to calculate variance

Consider

$$
\begin{aligned}
E\left[X^k\right] &= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1} (1-p)^{n-r} \\
&(\text{let } m = n+1, s = r+1) \\
&= \frac{r}{p} \sum_{m=s}^{\infty} (m-1)^{k-1} \binom{m-1}{s-1} p^s (1-p)^{m-s} \\
&= \frac{r}{p} E\left[(Y-1)^{k-1}\right]
\end{aligned}
$$

where Y is a negative binomial random variable with parameters $r+1, p$. Now let $k=2$.

Comparison of four discrete r.v.

Given we have an independent Bernoulli test in each time-slot.

- Binomial r.v. is the total count of success within some interval
- Poisson r.v. is limit of binomial distribution given mean of count of success fixed.
- Geometric r.v. is count of test until the first success
- Negative Binomial r.v. is the count of test until several success.

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Barking Dog

A dog barks with probability 0.01 each minute.

- How many times do we expect the dog to bark between noon and midnight?

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- It will be a geometric r.v..
- What is the probability the fifth bark since noon is at midnight?

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It will be expectation of negative binomial r.v..

Approximate the probability there are exactly 5 barks between noon and midnight.

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How many minutes do I expect to wait until the fifth bark?
It will be expectation of negative binomial r.v..

Approximate the probability there are exactly 5 barks between noon and midnight.
It will be Poisson r.v..