Math 18500 Week 8: Laplace Transforms

Laplace Transforms. We have seen that Fourier series and Fourier transforms give us an entirely different way of thinking about functions - in the *frequency domain* rather than in the *time domain.* However, they are not well-adapted to solving initial value problems, for reasons we have already discussed.

One method for thinking in the frequency domain in the context of initial value problems is to use the *Laplace transform.* The Laplace transform can be applied to any function $y(t)$ which is defined on the interval $[0, \infty)$, with the idea being that if one is interested in solving an initial value problem

$$my'' + by' + ky = f(t) \ , \ \ y(0) = y_0 \ , \ \ y'(0) = v_0,$$

then one is not interested in the values of $y(t)$ for $t < 0$.

When we take the Laplace transform of $y(t)$, the result is a new function of a different variable, $Y(s)$. This new function is defined as follows:

$$Y(s) = \int_0^\infty y(t) e^{-st} dt$$

Often, we will use the shorthand

$$Y(s) = \mathcal{L}\left\{y(t)\right\},$$

where $\mathcal{L}$ stands for "Laplace transform".

Most of the time, we will think of s as a real number, but it is important to be aware that it is possible to substitute complex values of s as well.

Notice that the Laplace transform has a similar appearance to the Fourier transform,

$$\hat{y}(k) = \int_{-\infty}^\infty y(t) e^{-ikt} dt.$$

In fact, *if we assume that the function $y(t)$ is zero for $t < 0$ and decays rapidly to zero as $t \to \infty$*, then we can recover $\hat{y}(k)$ from $Y(s)$ by making the substitution $s = ik$:

$$Y(ik) = \hat{y}(k)$$

For functions $y(t)$ satisfying this assumption, the Laplace transform can be viewed as a generalization of the Fourier transform, where ik is replaced with an arbitrary complex number s.

To get started computing Laplace transforms, let's find the Laplace transform of the function $y(t) = 1$:

$$\mathcal{L}\{1\} = \int_0^t 1 \cdot e^{-st} dt = -\frac{e^{-st}}{s}\bigg|_0^\infty = \left(\lim_{t\to\infty} \frac{e^{-st}}{s}\right) - \left(-\frac{1}{s}\right) = \frac{1}{s}.$$

Notice that this computation is only valid for $s > 0$, because we have made the assumption

$$\lim_{t\to\infty} e^{-st} = 0.$$

In general, the integral defining the Laplace transform will only converge for sufficiently large values of s (or more precisely, for complex values of s with sufficiently large real part).

For functions $f(t)$ which grow faster than any exponential function of the form e^{st}, such as

$$f(t) = e^{t^2},$$

the Laplace transform will not be defined for *any* value of s. More precisely, we must always assume that there is a value of s such that

$$\lim_{t\to\infty} f(t) e^{-st} = 0,$$

in order to legitimately take the Laplace transform of $f(t)$.

Next let's compute the Laplace transform of the function e^{3t}:

$$\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{3t} e^{-st} dt = \int_0^\infty e^{(3-s)t} dt = \frac{e^{(3-s)t}}{3-s}\bigg|_0^\infty = \left(\lim_{t\to\infty} \frac{e^{(3-s)t}}{3-s}\right) - \frac{1}{3-s} = \frac{1}{s-3}$$

Even though the function e^{3t} grows rapidly as $t \to \infty$, it still satisfies

$$\lim_{t\to\infty} e^{3t} e^{-st} = 0$$

as long as $\text{Re}[s] > 3$, and therefore it is legitimate to evaluate the Laplace transform for all such values of s.

In both examples we have done so far, the *formula* for the Laplace transform can be evaluated at any complex value of s (except for $s = 0$ in the first example, and $s = 3$ in the second example). This is a special case of a phenomenon called *analytic continuation*, which we will not discuss.

Both examples can be generalized as follows:

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at}e^{-st}dt = \frac{1}{s+a}.$$

This is valid for any value of a, and is worth memorizing.

As another example, we can compute the Laplace transform of a sinusoid, $y(t) = \cos(\omega t)$. To do this in the least painful way possible, we will exploit the fact that the Laplace transform is *linear*:

$$\mathcal{L}\{f(t) + g(t)\} = \int_0^\infty (f(t) + g(t))e^{-st}dt = \int_0^\infty f(t)e^{-st}dt + \int_0^\infty g(t)e^{-st}dt = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

This fact, combined with Euler's identity, gives us

$$\mathcal{L}\{\cos(\omega t)\} = \mathcal{L}\left\{\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right) = \frac{1}{2}\frac{s + i\omega + s - i\omega}{(s + i\omega)(s - i\omega)} = \frac{s}{s^2 + \omega^2}$$

A similar computation (try it yourself!) gives the Laplace transform of $\sin(\omega t)$:

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}.$$

In both cases, the integral defining the Laplace transform converges only when $s > 0$.

Now suppose we want to calculate the following Laplace transform:

$$\mathcal{L}\{e^{-3t}\cos(4t)\}$$

This can be done using a general rule about Laplace transforms, called the **exponential shift rule**. This rule states that if the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s),$$

then the Laplace transform of $e^{-at}f(t)$ is

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s+a)t}dt = F(s + a).$$

Applying this, we see that since the Laplace transform of $\cos(4t)$ is

$$\mathcal{L}\{\cos(4t)\} = \frac{s}{s^2 + 4^2},$$

the Laplace transform of $e^{-3t}\cos(4t)$ is

$$\mathcal{L}\{e^{-3t}\cos(4t)\} = \frac{s + 3}{(s + 3)^2 + 4^2}.$$

In general, the transform of an exponential times a sinusoid can be found using the following rules:

$$\mathcal{L}\{e^{-at}\cos(bt)\} = \frac{s + a}{(s + a)^2 + b^2}.$$

and

$$\mathcal{L}\{e^{-at}\sin(bt)\} = \frac{b}{(s + a)^2 + b^2}.$$

Another important rule for computing Laplace transforms is the rule for Laplace transforms of derivatives, which states that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

This can proved using integration by parts:

$$\begin{aligned}\int_0^\infty f'(t)e^{-st}dt &= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt \\ &= \left(\lim_{t\to\infty} f(t)e^{-st}\right) - f(0) + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + sF(s)\end{aligned}$$

Notice that for the computation above to be valid, we had to assume that

$$\lim_{t\to\infty} f(t)e^{-st} = 0,$$

for all sufficiently large values of s.

There is a similar rule for second derivatives, which can be obtained by applying the rule for first derivatives twice in a row:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

In general for higher derivatives of order n, we will always have

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

As an application of this rule, suppose we want to calculate the Laplace transform of the function

$$f(t) = \frac{t^n}{n!}$$

for some integer n. Since the n^{th} derivative of this function is

$$f^{(n)}(t) = 1,$$

and

$$f^{(k)}(0) = 0$$

for all $k < n$, we can conclude that

$$\frac{1}{s} = \mathcal{L}\{1\} = s^nF(s)$$

and therefore,

$$F(s) = \frac{1}{s^{n+1}}$$

This gives a general rule for Laplace transforms of powers of t:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

In combination with the exponential shift rule, this rule allows us to compute Laplace transforms of arbitrary functions of the form

$$f(t) = P(t)e^{at}$$

where $P(t)$ is a polynomial and e^{at} is any real or complex exponential.

Inverse Laplace Transforms. There is a mathematical theorem which states that any function is *uniquely determined* by its Laplace transform. In other words, if two functions have the same Laplace transform,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$$

Then the functions themselves are equal:

$$f(t) = g(t).$$

This result is a consequence of the Fourier inversion formula - you can read a proof in the appendix to this week's notes if you are interested.

For the kinds of functions $f(t)$ we have been considering up until now - linear combinations of exponentials times sinusoids - the Laplace transform will be a *rational function*, i.e. a ratio of two polynomial functions:

$$F(s) = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n}{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}$$

Given any such function, we can attempt to work backwards and figure out which function it is the Laplace transform of.[1] If we determine that

$$F(s) = \mathcal{L}\{f(t)\},$$

then we will write

$$f(t) = \mathcal{L}^{-1}(F(s)).$$

and say that $f(t)$ is the **Inverse Laplace transform** of $F(s)$.

There is a formula for the inverse Laplace transform, similarly to how there is a formula for the inverse Fourier transform. You can read about this formula in the appendix mentioned above. However, we will never use the formula, because it is too complicated! Instead, our method will always be to *recognize* the function $F(s)$ as the Laplace transform of a function we are familiar with.

A key property of the inverse Laplace transform is that it is *linear* - if

$$\mathcal{L}\{f(t)\} = F(s)\ ,\ \ \mathcal{L}\{g(t)\} = G(s),$$

then

$$\mathcal{L}\{f(t) + g(t)\} = F(s) + G(s),$$

and therefore,

$$\mathcal{L}^{-1}\{F(s) + G(s)\} = f(t) + g(t) = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}$$

In the examples which follow we will use this property without comment.

The inverse Laplace transform of a rational function can always be found using *partial fractions*. For example, consider the function

$$F(s) = \frac{2s+3}{s^2+3s-4}.$$

To find the inverse Laplace transform, we first factorize the denominator:

$$F(s) = \frac{2s+3}{(s-1)(s+4)}.$$

We then attempt to find a partial fractions decomposition:

$$\frac{2s+3}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4}$$

You may have seen how to do this by combining the denominators on the right hand side:

$$\frac{2s+3}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4} = \frac{A(s+4)+B(s-1)}{(s-1)(s+4)} = \frac{(A+B)s + (4A-B)}{(s-1)(s+4)}$$

In order for this to be valid, A and B must satisfy the system of equations

$$A + B = 2\ ,\ \ 4A - B = 3$$

which can be solved, giving $A = B = 1$.

[1]An inverse Laplace transform always exists provided $\lim_{s\to\infty} F(s) = 0$, or equivalently if $n < m$. If this condition does not hold, one must introduce the concept of a "generalized function" to describe the inverse Laplace transform.

But you should be aware that there is a far more efficient method, which doesn't involve solving systems of equations! It goes like this. To find A, we first multiply both sides by $s-1$:

$$\frac{2s+3}{s+4} = A + \frac{B(s-1)}{s+4}$$

If we set $s=1$, the term involving B vanishes, and we get

$$A = \frac{2\cdot 1+3}{1+4} = \frac{5}{5}1.$$

Similarly, if we first multiply by $s+4$,

$$\frac{2s+3}{(s-1)} = \frac{A(s+4)}{s-1} + B,$$

and then set $s=-4$, we get

$$B = \frac{2\cdot(-4)+3}{(-4)-1} = \frac{-5}{-5} = 1.$$

This is called the *Heaviside cover up method*. To get A, we cover up $s-1$, and to get B, we cover up $s+4$.

No matter how we find the values $A=1$ and $B=1$, we can conclude that

$$F(s) = \frac{1}{s-1} + \frac{1}{s+4}$$

Taking the inverse Laplace transform of both sides, we obtain the inverse Laplace transorm:

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^t + e^{-4t}.$$

In general, for any function of the form

$$F(s) = \frac{c_0 + c_1 s + \cdots + c_{n-1}s^{n-1}}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)},$$

where the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator, it is usually possible to find a partial fractions decomposition of the form

$$F(s) = \frac{A_1}{s-\lambda_1} + \frac{A_2}{s-\lambda_2} + \cdots + \frac{A_n}{s-\lambda_n}$$

using the method above. This gives us the inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \cdots + A_n e^{\lambda_n t}$$

However, there is a very important exception to this rule! If the denominator has *repeated roots*, then additional terms must be included. We will explain how this works later.

When the denominator has complex factors (but no repeated roots), we can do partial fractions with complex numbers (which is no big deal and works just fine!). However, there is also an alternate method, which is better adapted to obtaining inverse Laplace transforms in real form.

To get the idea of how the alternate method works, let's find the inverse Laplace transform of

$$G(s) = \frac{2s+3}{s^2+2s+2}.$$

As a first step we can "complete the square" in the denominator, by looking for values of a and b such that

$$s^2+2s+2 = (s+a)^2 + b^2 = s^2 + 2as + (a^2+b^2)$$

For this identity to be valid, we must have $2a=2$ and $a^2+b^2=1$, hence $a=1$ and $b=1$. We conclude that

$$s^2+2s+2 = (s+1)^2+1.$$

We can then try to write

$$G(s) = \frac{A+B(s+1)}{(s+1)^2+1}.$$

for constants A and B. In order for this to be valid we must have

$$A + B(s+1) = 2s+3,$$

which (after a bit of work) implies that $B = 2$ and $A = 1$, so

$$G(s) = \frac{1 + 2(s+1)}{(s+1)^2 + 1}.$$

Applying the linearity and exponential shift rules, we see that this is the Laplace transform of

$$g(t) = e^{-t}\sin t + 2e^{-t}\cos t,$$

and therefore,

$$\mathcal{L}^{-1}\{G(s)\} = e^{-t}\sin t + 2e^{-t}\cos t.$$

In general, given a rational function with a combination of real and complex factors in its denominator, like

$$H(s) = \frac{2s+3}{(s+3)(s^2+2s+2)} = \frac{s^2+2s+3}{(s+3)((s+1)^2+1)}$$

we can seek a partial fractions decomposition of the form

$$H(s) = \frac{A}{s+3} + \frac{B + C(s+1)}{(s+1)^2+1}.$$

with one term for each real root of the denominator and one factor for each pair of complex conjugate roots.

We know that it is possible to find such a decomposition, because we can always do a complex partial fractions decomposition and write

$$H(s) = \frac{A}{s+3} + \frac{Z}{s-(1+i)} + \frac{W}{s-(1-i)}.$$

for complex numbers Z and W. Combining the last two terms yields an expression of the form

$$\frac{Z}{s-(1+i)} + \frac{W}{s-(1-i)} = \frac{(Z+W)s + i(Z-W)}{(s-(1+i))(s-(1-i))} = \frac{B + C(s+1)}{(s+1)^2+1}.$$

Since $H(s)$ is real, the numbers B and C must be real (and Z and W must be complex conjugates).

So the real question is - what is the most efficient way to determine the coefficients A, B, and C? Of course, A can be found using the cover-up method. We multiply by $s+3$,

$$\frac{2s+3}{s^2+2s+2} = A + (s+3)\frac{B + C(s+1)}{(s+1)^2+1}$$

and set $s = -3$, which gives us

$$A = \frac{-2\cdot 3 + 3}{(-3)^2 - 2\cdot 3 + 2} = -\frac{3}{5}$$

To find B and C we can apply a similar trick - just multiply both sides by $(s+1)^2 + 1$:

$$\frac{2s+3}{s+3} = ((s+1)^2+1)\frac{A}{s+3} + B + C(s+1)$$

We can then substitute $s = -1 + i$, which is a root of the equation

$$(s+1)^2 + 1 = 0.$$

This gives us

$$B + Ci = \frac{-2 + 2i + 3}{-1 + i + 3} = \frac{1+2i}{2+i} = \frac{(1+2i)(2-i)}{5} = \frac{4+3i}{5}$$

Since B and C are real, we must have $B = \frac{4}{5}$ and $C = \frac{3}{5}$. Putting it all together, we find that

$$H(s) = \frac{2s+3}{(s+3)(s^2+2s+2)} = -\frac{3}{5}\cdot\frac{1}{s+3} + \frac{1}{5}\cdot\frac{3(s+1)+4}{(s+1)^2+1}$$

This method allows you to find inverse Laplace transforms of general rational functions with both real and complex roots, provided that there are no repeated roots.

Now we are ready to discuss the case where the denominator has repeated roots. Suppose we want to find a partial fractions decomposition of the function

$$K(s) = \frac{s^2 - s}{s^2(s+1)}.$$

In this case, it is not possible find constants A and B such that

$$\frac{s^2 - s}{s^2(s+1)} = \frac{A}{s+1} + \frac{B}{s},$$

because combining the fractions on the right hand side would give

$$\frac{As + B(s+1)}{s(s+1)}$$

and the denominator of this function is not $s^2(s+1)$.

It is also not possible to find constants A and C such that

$$\frac{s^2 - s}{s^2(s+1)} = \frac{A}{s+1} + \frac{C}{s^2},$$

because combining the fractions would give

$$\frac{As^2 + C(s+1)}{s^2(s+1)}$$

and there are no values of A and C such that

$$As^2 + C(s+1) = s^2 - s.$$

What does work is to seek a partial fractions decomposition of the form

$$\frac{s^2 - s}{s^2(s+1)} = \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s^2}$$

In this case combining the fractions gives

$$\frac{s^2 - s}{s^2(s+1)} = \frac{As^2 + Bs(s+1) + C(s+1)}{s^2(s+1)} = \frac{(A+B)s^2 + (B+C)s + C}{s^2(s+1)}$$

and this gives the system of equations

$$A + B = 1\ , \quad B + C = -1\ , \quad C = 0$$

which does have a solution,

$$A = 2\ , \quad B = 1\ , \quad C = 0$$

You might ask whether there is a generalization of the cover-up method which will allow us to efficiently find A, B, and C without solving a system of equations. The answer is "sort of". We can certainly find A by multiplying through by $s+1$:

$$\frac{s^2 - s}{s^2} = A + (s+1)\left(\frac{B}{s} + \frac{C}{s^2}\right)$$

Then setting $s = -1$ gives us $A = -2$.

We can also find C by multiplying through by s^2:

$$\frac{s^2 - s}{s+1} = \frac{As^2}{s+1} + Bs + C$$

Then setting $s = 0$ gives us the value $C = 0$.

Unfortunately, there is not a similar method for determining B. One trick is to set s equal to a completely different value, which is not a root of the denominator. For example, setting $s = 1$ gives

$$\frac{1^2 - 1}{1^2(1+1)} = \frac{-2}{1+1} + \frac{B}{1} + \frac{0}{1}$$

and solving for B we obtain the value

$$B = 1$$

This trick will work as long as there is only one repeated root with multiplicity 2. For a root which is repeated m times, you must add an entire series of terms,

$$\frac{A_1}{(s-\lambda)} + \frac{A_2}{(s-\lambda)^2} + \cdots + \frac{A_m}{(s-\lambda)^m}$$

where m is the multiplicity of the root. Unfortunately, in cases where the multiplicity is greater than 2, or when there is more than one repeated root, determining the values of A_1, A_2, $\ldots A_{m-1}$ will require you to solve a system of equations.[2]

It is worth observing what the inverse Laplace transform of the above expression is. Since

$$\mathcal{L}\{t^{k-1}\} = \frac{(k-1)!}{s^k}$$

we can apply linearity and the exponential shift rule and conclude that

$$\mathcal{L}^{-1}\left\{\frac{A_k}{(s-\lambda)^k}\right\} = A_k \frac{t^{k-1}}{(k-1)!} e^{\lambda t}.$$

[2]Actually, there is a trick: after multiplying through by $(s-\lambda)^m$, you can take the derivative k times and then set $s = \lambda$. In theory, this will give you the value of A_{m-k}. But in practice you may find that "the cure is worse than the disease".

Transforms of Initial Value Problems. We will now see how to use Laplace transforms to solve initial value problems of the form

$$my'' + ly' + ky = f(t), \;\; y(0) = y_0, \;\; y'(0) = v_0$$

The general strategy is to first take the Laplace transform of both sides of the equation,

$$m\mathcal{L}\{y''\} + l\mathcal{L}\{y'\} + k\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}.$$

We then attempt to solve this equation for the Laplace transform of $y(t)$,

$$Y(s) = \mathcal{L}\{y(t)\}$$

Once we have $Y(s)$, we can take the inverse Laplace transform to obtain $y(t)$!

As you will see, the main appeal of this method is that solving for $Y(s)$ is pure algebra. The Laplace transform converts *differential* equations into *algebraic* equations.

As a simple example, let's solve the initial value problem

$$y'' + 2y' + y = e^{-t}, \;\; y(0) = y'(0) = 0.$$

Taking the Laplace transform of both sides, we can solve for $Y(s)$:

$$\begin{aligned}(s^2Y(s) - sy'(0) - y(0)) + 2(sY(s) - y(0)) + Y(s) &= \frac{1}{s+1}\\ (s+1)^2Y(s) &= \frac{1}{s+1}\\ Y(s) &= \frac{1}{(s+1)^3}\end{aligned}$$

Taking the inverse Laplace transform, we get the solution,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}e^{-t} = \frac{t^2}{2}e^{-t}$$

For a more involved example, consider the initial value problem

$$y'' + 4y' + 8y = e^{3t}, \;\; y(0) = 1, \;\; y'(0) = -5.$$

To do this, we take the Laplace transform of both sides:

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 8Y(s) = \frac{1}{s-3}$$

Substituting our chosen initial values, we find that

$$(s^2 + 4s + 8)Y(s) - s + 1 = \frac{1}{s-3}$$

and solving for $Y(s)$ gives

$$Y(s) = \frac{1}{(s^2+4s+8)(s-3)} + \frac{1-s}{s^2+4s+8}$$

This part is always easy - solving for $Y(s)$ is always just simple algebra!

The hard part is finding the inverse Laplace transform of $Y(s)$. Since the equation

$$s^2 + 4s + 8 = (s+2)^2 + 2^2 = 0$$

has complex roots $s = -2 \pm 2i$, we seek a partial fractions decomposition of the form

$$Y(s) = \frac{1}{((s+2)^2+2^2)(s-3)} + \frac{1-s}{(s+2)^2+2^2} = \frac{A + B(s+2)}{(s+2)^2+2^2} + \frac{C}{s-3}.$$

Again, this can be done using the cover-up method. To find C, we multiply on both sides by $s - 3$:

$$\frac{1}{(s+2)^2+2^2} + \frac{(1-s)(s-3)}{(s+2)^2+2^2} = \frac{(A + B(s+2))(s-3)}{(s+2)^2+2^2} + C$$

Setting $s = 3$, we see that all but the leftmost and rightmost terms vanish, and we obtain

$$C = \frac{1}{(3+2)^2+2^2} = \frac{1}{29}$$

To find A and B, we first multiply on both sides by $(s+2)^2+2^2$:

$$\frac{1}{s-3}+1-s=A+B(s+2)+C\frac{(s+2)^2+2^2}{s-3}$$

To get rid of the term involving C, we must set s equal to a root of the equation

$$(s+2)^2+2^2=0$$

e.g. $s=-2+2i$. Doing this, we get

$$\frac{1}{-5+2i}+-3-2i=A+B(-2+2i+2)$$
$$\frac{-5-2i}{29}-3-2i=A+2Bi$$

Taking real and imaginary parts of both sides, we get

$$A=\frac{-5}{29}-3=\frac{-92}{29}$$
$$B=\frac{-1}{29}-1=\frac{-30}{29}.$$

Therefore,

$$Y(s)=-\frac{92}{29}\frac{1}{(s+2)^2+2^2}-\frac{30}{29}\frac{s+2}{(s+2)^2+2^2}+\frac{1}{29}\frac{1}{s-3},$$

and

$$y(t)=\mathcal{L}^{-1}\left\{Y(s)\right\}=-\frac{92}{29}\cdot\frac{1}{2}e^{-2t}\sin(2t)-\frac{30}{29}e^{-2t}\cos(2t)+\frac{1}{29}e^{3t}.$$

Technical Appendix. In this section we will describe in more detail how the Laplace transform and Fourier transform are related, and give a formula for the inverse Laplace transform.

Let $f_+(t)$ be a function which is defined on the interval $[0, \infty)$. Then it can be extended to a function on the real line by defining

$$f(t) = \begin{cases} 0 & t < 0 \\ f_+(t) & t > 0 \end{cases}$$

From the point of view of the Laplace transform, there is no difference between $f_+(t)$ and $f(t)$, because

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty f_+(t)e^{-st}dt = \mathcal{L}\{f_+(t)\}$$

Now assume that there is a value of s such that

$$\int_{-\infty}^{\infty} |f(t)e^{-st}|^2 dt < \infty$$

i.e. such that the function

$$g(t) = f(t)e^{-st}$$

dies off very rapidly as $t \to \infty$. Then it is permissible to apply the Fourier inversion formula to the function $g(t)$. This gives us

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(k)e^{ikt}dt = g(t) = f(t)e^{-st}$$

where

$$\hat{g}(k) = \int_{-\infty}^{\infty} f(t)e^{-s_0 t}e^{-ikt}dt = F(s + ik)$$

This gives us a formula for the inverse Laplace transform:

$$f(t) = e^{st}\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s + ik)e^{ikt}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s + ik)e^{(s+ik)t}dk$$

and proves that the function $f(t)$ is uniquely determined by its Laplace transform.