Lecture 6: Euler Approximation

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Abstract

Backward and forward representation, strong and weak convergence of Euler approximation.

1 Backward and Forward Representations

Let $X(t)$ be a diffusion process (solution of SDE) with drift $a(t,x)$, diffusion $b(t,x)$:

$$dX_t = adt + bdW_t,$$

consider the conditional expectation ($s < t$):

$$E(f(X_t)|X_s = x) = \int f(y)\, p(s,x;t,y)\, dy, \tag{1.1}$$

where $p(s,x;t,y)$ is the transition probability density function from (s,x) to (t,y). As a function of (s,x), p satisfies the *backward equation*:

$$p_s + \frac{1}{2}b^2(s,x)p_{xx} + a(s,x)p_x = 0. \tag{1.2}$$

Hence $u(s,x) = E(f(X_t)|X_s = x)$ solves (1.2) with final condition $u(t,x) = f(x)$.
For the forward representation, consider the *Autonomous case*, $a = a(x)$, $b = b(x)$. Then $p(s,t;x,y) = p(t-s;x,y)$, $p_s = -p_t$,

$$p_t = \frac{1}{2}b^2(x)p_{xx} + a(x)p_x,\ t > s, \tag{1.3}$$

$p(t;x,y) \to \delta(y-x)$, as $t \to 0+$. The transition probability density becomes fundamental solution of parabolic equation (1.3). As a function of (t,x),

$$v(t,x) = E(f(X_t)|X_s = x), \tag{1.4}$$

solves:

$$v_t = \frac{1}{2}b^2(x)v_{xx} + a(x)v_x, \tag{1.5}$$

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with initial data $v(s,x) = f(x)$.
Eq. (1.4) is a probabilistic representation formula of PDE (1.5). It can be generalized to include a lower order (potential) term as in Eqn:

$$w_t = \frac{1}{2} b^2(x) w_{xx} + a(x) w_x + V(x) w,\ t > 0, \tag{1.6}$$

initial data: $w(0,x) = f(x)$. The Feynmann-Kac formula is:

$$w(t,x) = E\left[\exp\{\int_0^t V(X(\tau))\, d\tau\} f(X(\tau))\right], \tag{1.7}$$

If the diffusion $b(x) \equiv 0$, F-K formula reduces to a solution formula of first order hyperbolic eqn by the method of characteristics.
To derive (1.7), let:

$$T_t f = E\left[\exp\{\int_0^t V(X(\tau))\, d\tau\} f(X(\tau))\right],$$

a linear bounded (nonnegative) operator on the space of bounded continuous functions. Note:

$$\exp\{\int_0^t V(X_s) ds\} = 1 + \int_0^t V(X_s) ds + o(t),$$

as $t \to 0+$. We have for any $f(x)$ in the domain of T_t:

$$\begin{aligned} \frac{T_t f(x) - f(x)}{t} &= \frac{1}{t}\left(E[f(X_t) e^{\int_0^t V(X_s) ds}] - f(x)\right) \\ &= \frac{1}{t}(E[f(X_t)] - f(x)) + \frac{1}{t} E[f(X_t) \int_0^t V(X_s) ds] \\ &\to (b^2(x) f_{xx}/2 + a(x) f_x) + V(x) f. \end{aligned} \tag{1.8}$$

We have used (1.3) for the limit of first term.
To generalize F-K to nonautonomous case, treat t as a parameter,

$$\begin{aligned} dX_s^{t,x} &= a(t_s^{t,x}, X_s^{t,x}) ds + b(t_s^{t,x}, X_s^{t,x}) dW_s, \\ dt_s^{t,x} &= -ds, \end{aligned} \tag{1.9}$$

$X_0^{t,x} = x$, $t_0^{t,x} = t$, symmetrically extending a, b: $a(-\tau, x) = a(\tau, x)$ etc. *View (1.9) as a diffusion process on* $(t,x) \in R^2$ *with time* s. Eqs (1.9) are autonomous, and define a Markov process $(t_s^{t,x}, X_s^{t,x}, P)$. We then apply F-K (1.7). The result is:

$$w(t,x) = E f(X_t^{t,x}) \exp\{\left[\int_0^t V(t-s, X_s^{t,x}) ds\right]\}, \tag{1.10}$$

solves eqn:

$$w_t = \frac{1}{2} b^2(t,x) w_{xx} + a(t,x) w_x + V(t,x) w, \tag{1.11}$$

$w(0,x) = f(x)$.

All results generalize to higher space dimensions.

2 Euler Method: Order of Convergence

SDE:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t,\ t \in (0, T], \tag{2.12}$$

with initial value X_0 at $t = 0$. Discrete times $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N = T$. Denote $\Delta_n = t_{n+1} - t_n$, $\delta = \max \Delta_n$.
Euler approximation:

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta_n + b(t_n, Y_n)(W_{t_{n+1}} - W_{t_n}), \tag{2.13}$$

with $Y_0 = X_0$.
Y_n is A_n measurable.
Connecting the adjacent discrete values Y_n by straight lines form a continuous function $Y(t)$. Pathwise measure of approximation is:

$$\epsilon = E(|X(T) - Y(T)|), \tag{2.14}$$

reduces to deterministic absolute error at $t = T$ if noise is absent. In actual computation, suppose we have N solutions $Y_k(T)$ from N realizations of BM, then ϵ is approximated by:

$$\tilde{\epsilon} = \frac{1}{N} \sum_{k=1}^{N} |X_k(T) - Y_k(T)|. \tag{2.15}$$

It is an amusing fact that $\epsilon \sim O(\delta^{1/2})$ in the stochastic case while $\epsilon \sim O(\delta)$ in the deterministic case. This is analyzed below.

2.1 Strong Convergence/Consistency

Strong Convergence if:

$$\lim_{\delta \to 0} E(|X(T) - Y_\delta(T)|) = 0. \tag{2.16}$$

Strong Convergence with order $\gamma > 0$:

$$E(|X(T) - Y_\delta(T)|) \le C\delta^\gamma, \tag{2.17}$$

for any $\delta \in [0, \delta_0]$, $\delta_0 > 0$.
Strong Consistency of Discrete Approximation:

$$E\left(|E(\frac{Y^\delta_{n+1} - Y^\delta_n}{\Delta_n} | A_{t_n}) - a(t_n, Y^\delta_n)|^2 \right) \le c(\delta) \to 0, \tag{2.18}$$

and:

$$\begin{aligned} &E\left[\frac{1}{\Delta_n} |Y^\delta_{n+1} - Y^\delta_n - E(Y^\delta_{n+1} - Y^\delta_n | A_{t_n}) - b(t_n, Y^\delta_n)\Delta W_n|^2 \right] \\ &\le c(\delta) \to 0. \end{aligned} \tag{2.19}$$

for all fixed $Y^\delta_n = y$, $n = 0, 1, 2, \cdots$.
For Euler, strong consistency holds with $c(\delta) \equiv 0$.

2.2 Convergence

Consider the autonomous SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{2.20}$$

Theorem 2.1 *A strongly consistent equidistant time discrete approximation Y^δ_n of (2.20) with $Y^\delta(0) = X_0$ converges strongly to X. In particular, Euler method converges strongly with order 1/2.*

Sketch of Proof:

$$Z(t) = \sup_{s\in[0,t]} E(|Y^\delta_{n_s} - X(s)|^2),$$

$$n_s = \max\{n : t_n \le s\}.$$

$$Z(t) = \sup_{s\in[0,t]} E[|\sum_{n=0}^{n_s-1}(Y^\delta_{n+1} - Y^\delta_n) - \int_0^s a(X_r)ds - \int_0^s b(X_r)dW_r|^2]$$

$$\begin{aligned}
&\le C_1 \sup_{s\in[0,t]} \Big\{E[|\sum_{n=0}^{n_s-1}(E(Y^\delta_{n+1} - Y^\delta_n|A_{t_n}) - a(Y^\delta_n)\Delta_n)|^2] \\
&+ E[|\sum_{n=0}^{n_s-1}(Y^\delta_{n+1} - Y^\delta_n - E(Y^\delta_{n+1} - Y^\delta_n|A_{t_n}) - b(Y^\delta_n)\Delta W_n)|^2] \\
&+ E[|\int_0^{t_{n_s}} a(Y^\delta_{n_r}) - a(X_r)\,dr|^2] + E[|\int_0^{t_{n_s}} b(Y^\delta_{n_r}) - b(X_r)\,dW_r|^2] \\
&+ E[|\int_{t_{n_s}}^s a(X_s)ds|^2] + E[|\int_{t_{n_s}}^s b(X_s)dW_s|^2]\Big\}
\end{aligned} \tag{2.21}$$

by strong consistency and Lipschitz condition:

$$Z(t) \le C_2 \int_0^t Z(s)ds + C_3(\delta + c(\delta)), \tag{2.22}$$

the last term of (2.21) contributes $O(\delta)$.
Gronwall inequality:

$$Z(t) \le C_4(\delta + c(\delta)), \tag{2.23}$$

or:

$$E(|Y^\delta(T) - X(T)|) \le C_5\sqrt{\delta + c(\delta)}, \tag{2.24}$$

strong convergence. For Euler, $c(\delta) = 0$, $\gamma = 1/2$.

3 Weak Consistency: Definition and Examples

A discrete SDE approximation $Y^{\delta}(t)$ is called *converging weakly* to $X(t)$ at $t = T$ if:

$$\lim_{\delta \to 0} |E(g(X(T))) - E(g(Y^{\delta}(T)))| = 0, \tag{3.25}$$

for any $g \in \mathcal{C}$, $\mathcal{C}$ a class of smooth test functions. One example of $\mathcal{C}$ is all polynomials, then (3.25) is same as convergence of all moments of solutions. As before, discrete times $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N = T$, $\Delta_n = t_{n+1} - t_n$, $\delta = \max \Delta_n$.
Convergence is order $\beta > 0$ if:

$$|E(g(X(T))) - E(g(Y^{\delta}(T)))| \leq C\delta^{\beta}, \tag{3.26}$$

for small δ.
Later we will see that Euler method is weakly convergent of order $\beta = 1$, while it is order 1/2 strong convergent (pathwise).

The discrete approximation is *weakly consistent* if

$$E\left(\left| E\left(\frac{Y^{\delta}_{n+1} - Y^{\delta}_{n}}{\Delta_n} | A_{t_n} \right) - a(t_n, Y^{\delta}_n) \right|^2 \right) \leq c(\delta) \to 0, \tag{3.27}$$

same as in strong consistency, and:

$$E\left[\left| E\left(\frac{1}{\Delta_n} (Y^{\delta}_{n+1} - Y^{\delta}_{n})^2 | A_n \right) - b^2(t_n, Y^{\delta}_n) \right|^2 \right]$$
$$\leq c(\delta) \to 0. \tag{3.28}$$

for all fixed $Y^{\delta}_n = y$, $n = 0, 1, 2, \cdots$.
For Euler, weak consistency holds. Moreover, some modified Euler like:

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta_n + b(t_n, Y_n)\xi_n (\Delta_n)^{1/2}, \tag{3.29}$$

where ξ_n independent two point r.v., $P(\xi_n = \pm 1) = 1/2$, is weakly convergent, not strongly convergent.

4 Consistency implies Convergence

Consider the autonomous SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{4.30}$$

a, b, smooth, with polynomial growth.

Theorem 4.1 *Consider equidistant time weakly consistent discrete approximation Y_n^δ of (4.30) with $Y^\delta(0) = X_0$ so that:*

$$E(\max_n |Y_n^\delta|^{2q}) \leq K(1 + E(|X_0|^{2q})), \tag{4.31}$$

for $q = 1, 2, \cdots$, and:

$$E(|Y_{n+1}^\delta - Y_n^\delta|^6) \leq c(\delta)\Delta_n, \quad c(\delta) = o(\delta), \tag{4.32}$$

for any $n = 0, 1, 2, \cdots$. Then Y_n^δ converges weakly to $X(t)$.

Sketch of Proof: Write $Y(t) = Y^\delta(t)$.
Use fact:

$$u(s, x) = E(g(X_T)|X_s = x), \tag{4.33}$$

solves backward equation:

$$u_s + Lu = u_s + au_x + \frac{b^2}{2}u_{xx} = 0, \tag{4.34}$$

and:

$$u(T, x) = g(x). \tag{4.35}$$

Denote by $X_t^{s,x}$ solution of:

$$X_t^{s,x} = x + \int_s^t a(X_r^{s,x})dr + \int_s^t b(X_r^{s,x})dW_r. \tag{4.36}$$

Ito formula and (4.34) give:

$$E(u(t_{n+1}, X_{t_{n+1}}^{t_n,x}) - u(t_n, x)|A_n) = 0, \tag{4.37}$$

By eqns (4.33)-(4.35), write:

$$\begin{aligned} H &= |E(g(Y(T))) - E(g(X(T)))| \\ &= |E(u(T, Y(T)) - u(0, Y_0))| \\ &= |E(\sum_{n=0}^{n_T-1} u(t_{n+1}, Y_{n+1}) - u(t_n, Y_n))|. \end{aligned} \tag{4.38}$$

By (4.37):

$$\begin{aligned} H &= |E(\sum[u(t_{n+1}, Y_{n+1}) - u(t_n, Y_n) \\ &\quad -(u(t_{n+1}, X_{t_{n+1}}^{t_n,Y_n}) - u(t_n, X_{t_n}^{t_n,Y_n}))])| \\ &= |E(\sum[u(t_{n+1}, Y_{n+1}) - u(t_{n+1}, Y_n) \\ &\quad -(u(t_{n+1}, X_{t_{n+1}}^{t_n,Y_n}) - u(t_{n+1}, Y_n))])| \end{aligned}$$

Taylor expand in x:

$$\begin{aligned} H &= |E(\sum u_x[(Y_{n+1}-Y_n)-(X_{t_{n+1}}^{t_n,Y_n}-Y_n)] \\ &\quad +\frac{1}{2}u_{xx}[(Y_{n+1}-Y_n)^2-(X_{t_{n+1}}^{t_n,Y_n}-Y_n)^2] \\ &\quad +O(|Y_{n+1}-Y_n|^3+|X_{t_{n+1}}^{t_n,Y_n}-Y_n|^3))| \end{aligned} \tag{4.39}$$

u_x, u_{xx} evaluated at (t_{n+1}, Y_n).

Higher Moments Estimate of SDE (augmented, Theorem 4.5.4 in KL's book)
Suppose that conditions in lecture 5 hold and that

$$E\left(|X_{t_0}|^{2n}\right) < \infty$$

for some integer $n \geq 1$. Then the solution X_t satisfies

$$E\left(|X_t|^{2n}\right) \leq \left(1+E\left(|X_{t_0}|^{2n}\right)\right) e^{C(t-t_0)}$$

and

$$E\left(|X_t - X_{t_0}|^{2n}\right) \leq D\left(1+E\left(|X_{t_0}|^{2n}\right)\right)(t-t_0)^n e^{C(t-t_0)}$$

$$\begin{aligned} H &\leq C \sum E(|u_x||E((Y_{n+1}-Y_n)-(X_{t_{n+1}}^{t_n,Y_n}-Y_n)|A_n)| \\ &+ \frac{1}{2}|u_{xx}||E((Y_{n+1}-Y_n)^2-(X_{t_{n+1}}^{t_n,Y_n}-Y_n)^2|A_n)| \\ &\quad +O(\delta^{3/2}+\delta^{1/2}\sqrt{c(\delta)}) \\ &\leq C\sum \delta E^{1/2}(|E(\frac{Y_{n+1}-Y_n}{\delta}|A_n)-a(t_n,Y_n)|^2) \\ &+ E^{1/2}(|E(\frac{(Y_{n+1}-Y_n)^2}{\delta}|A_n)-b^2(t_n,Y_n)|^2) \\ &+ O(\delta^{3/2}+\delta^{1/2}\sqrt{c(\delta)}) \\ &\leq C\sum \delta\sqrt{c(\delta)}+O(\delta^{3/2}+\delta^{1/2}\sqrt{c(\delta)}) \\ &= O(\sqrt{c(\delta)}+\delta^{1/2}+\sqrt{c(\delta)/\delta}) \to 0. \end{aligned} \tag{4.40}$$