Math 18500 Week 2: First Order Differential Equations Part II.

First Order Linear Equations. A first order linear equation is an equation of the form

$$\frac{dy}{dx} = p(x)y + q(x)$$

where $p = p(x)$ and $q = q(x)$ are arbitrary functions of x.

We have already encountered one equation of this form - Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - T_0) = -kT + kT_0.$$

In this case the coefficients $p = -k$ and $q = kT_0$ were *constants* - this made the equation *separable*.

In the general case we need a completely different method. This method has four steps:

(1) Group all terms involving y on one side of the equation:

$$\frac{dy}{dx} - p(x)y = q(x),$$

and multiply both sides of the equation by an unspecified function $J = J(x)$:

$$J\frac{dy}{dx} - Jp(x)y = Jq(x)$$

This function is called the *integrating factor*.

(2) Choose the function J so that

$$\frac{dJ}{dx} = -Jp(x).$$

Any *nonzero* function J which satisfies this equation is OK. To find J, either separate variables:

$$\frac{dJ}{J} = -p(x)dx$$

or guess and check (there's no need to be systematic, only one solution is necessary).

(3) Replace $-Jp(x)$ with $\frac{dJ}{dx}$:

$$J\frac{dy}{dx} + \frac{dJ}{dx}y = Jq(x)$$

and apply the product rule in reverse:

$$\frac{d}{dx}[Jy] = Jq(x).$$

This explains why we needed to choose J the way we did in step (2).

(4) Integrate both sides and solve for y:

$$Jy = \int Jq(x)dx + C$$

$$y = \frac{1}{J}\left(\int Jq(x)dx + C\right)$$

When you apply this method, don't just use the formula at the end, go through all the steps! If you make a mistake in calculating J, it will become apparent in step 3, because you won't be able to apply the product rule in reverse. If you skip to the end, you won't notice the mistake, and your answer will be nonsense.

As an example, let's solve the equation

$$\frac{dT}{dt} = 1 - T.$$

Moving all instances of T to the left hand side, we obtain

$$\frac{dT}{dt} + T = 1.$$

Multiplying by our integrating factor we obtain

$$J\frac{dT}{dt} + JT = J.$$

In order to apply the product rule in reverse, we must choose J so that

$$\frac{dJ}{dt} = J.$$

In this case it's easiest to just guess a solution, instead of separating variables:

$$J = e^t$$

Substituting in our original equation, we have

$$e^t \frac{dT}{dt} + e^t T = e^t$$

Applying the product rule in reverse gives

$$\frac{d}{dt}\left[e^t T\right] = e^t$$

Integrating, we obtain

$$e^t T = e^t + C$$

so the general solution of the equation is

$$T = 1 + Ce^{-t},$$

the same as we obtained previously by separating variables.

Remember that when we separated variables, we arrived at this result by considering three separate cases, depending on whether C was positive, negative or 0. With the method of integrating factors there is much less fuss - we directly saw that C could be an arbitrary constant.

As a second example, let's solve the initial value problem

$$\frac{dy}{dx} = -\frac{y}{x} + e^{-x^2}\ , \quad y(1) = 0.$$

To solve this, we move all terms involving y to one side,

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x^2}$$

Multiply by J,

$$J\frac{dy}{dx} + J\frac{y}{x} = Je^{-x^2}$$

Set up a differential equation for J,

$$\frac{dJ}{dx} = \frac{J}{x} \implies \frac{dJ}{J} = \frac{dx}{x}$$

Notice that $J = x$ is a solution of the equation, and substitute:

$$x\frac{dy}{dx} + y = xe^{-x^2}$$

Apply the product rule in reverse,

$$\frac{d}{dx}\left[xy\right] = xe^{-x^2}$$

Integrate,

$$xy = -\frac{e^{-x^2}}{2} + C$$

and solve for y:

$$y = -\frac{e^{-x^2}}{2x} + \frac{C}{x}.$$

Finally, we determine the correct value of C, by substituting the initial conditions $x = 1$ and $y = 0$:

$$0 = -\frac{e^{-1}}{2} + \frac{C}{1} \implies C = \frac{e^{-1}}{2}$$

This gives us the solution:

$$y = -\frac{e^{-x^2}}{2x} + \frac{e^{-1}}{2x}.$$

Euler's Method. [*Optional section - read if you want to know how computers solve differential equations!*]

There are a few more methods for solving first order equations, which you can read about in your textbook (section 8.4). Some of these methods may become useful if you end up solving a lot of differential equations by hand, but we won't require you to know them in this course.

If you start looking at all of these different methods, you might be troubled by the fact that there doesn't seem to be a truly *general* method of solving first order equations - only special types of equations that require special methods of solution.

In this section and the next we will explain two methods of finding solutions which *are* truly general. But of course there is a catch: while these methods always work in *theory*, in most cases they will only lead to *approximations* of the actual solutions.

The first method of producing approximate solutions that we will describe is called *Euler's Method*. This method is a very basic version of the technique that computers use to solve differential equations.

Back in the 18th century, when Euler invented the method we're about to describe, the word "computer" had a different meaning: it was literally a person whose job was to do computations. These days, electronic computers use much more sophisticated methods to solve differential equations than Euler's method. If you ever actually need to write a computer program to solve differential equations, you should read about them (a good starting point is to google the phrase "Runge-Kutta").

The idea of Euler's method is to take the given differential equation or initial value problem,

$$y' = f(x, y) \ , \ \ y(x_0) = y_0$$

and *discretize* it. That is, instead of trying to compute the solution $y = y(x)$ as a *continuous* function of x, you instead try to calculate a sequence of values,

$$y(x_0) \ , \ \ y(x_0 + h) \ , \ \ y(x_0 + 2h) \ , \ \ y(x_0 + 3h) \ , \ \ldots$$

where h is a fixed small number, called the *step size*.

One way to estimate this sequence of values is to apply the linear approximation formula,

$$y(x + h) \approx y(x) + hy'(x).$$

If $y(x)$ is a solution of the equation

$$y'(x) = f(x, y),$$

then we can substitute for $y'(x)$:

$$y(x + h) \approx y(x) + hf(x, y(x)).$$

This equation allows us to *approximately* solve for $y(x_0 + nh)$, given the initial value $y(x_0)$:

$$\begin{array}{rcl} y(x_0) & = & y_0 \\ y(x_0 + h) & \approx & y(x_0) + hf(x_0, y(x_0)) \\ y(x_0 + 2h) & \approx & y(x_0 + h) + hf(x_0 + h, y(x_0 + h)) \\ y(x_0 + 3h) & \approx & y(x_0 + 2h) + hf(x_0 + 2h, y(x_0 + 2h)) \\ & \ldots & \end{array}$$

Typically these approximations get worse and worse the more steps we carry out (the errors *compound*). But with a very small value for h and a large number of steps, you can obtain decently accurate approximations.

For example, consider the initial value problem

$$y' = y^2 - x \ , \ \ y(0) = 1$$

In this case, the linear approximation formula tells us that

$$y(x + h) \approx y(x) + h(y(x)^2 - x).$$

Applying Euler's method with the step size $h = 0.1$, we obtain the following approximate values:

$$\begin{aligned} y(0) &= 1 \\ y(0.1) &\approx y(0) + 0.1 \cdot (y(0)^2 - 0) = 1 + 0.1 \cdot (1^2 - 0) = 1.1 \\ y(0.2) &\approx y(0.1) + 0.1 \cdot (y(0.1)^2 - 0.1) = 1.1 + 0.1 \cdot (1.1^2 - 0.1) = 1.211 \\ y(0.3) &\approx y(0.2) + 0.1 \cdot (y(0.2)^2 - 0.2) = 1.211 + 0.1 \cdot (1.211^2 - 0.2) = 1.337... \\ &\dots \end{aligned}$$

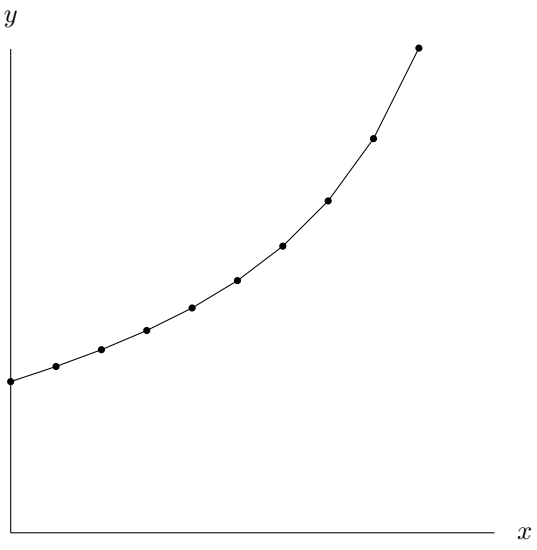
This quickly becomes something you don't want to do by hand, but you can make your computer do it! For example, here is a simple program (in the Python programming language) that implements Euler's method in the above example, with 9 steps of size 0.1:

```
x,y,h,N = 0,1,0.1,9
for i in range(N):
   y += h*(y**2-x)
   x += h
   print(round(x,3),round(y,3))
```

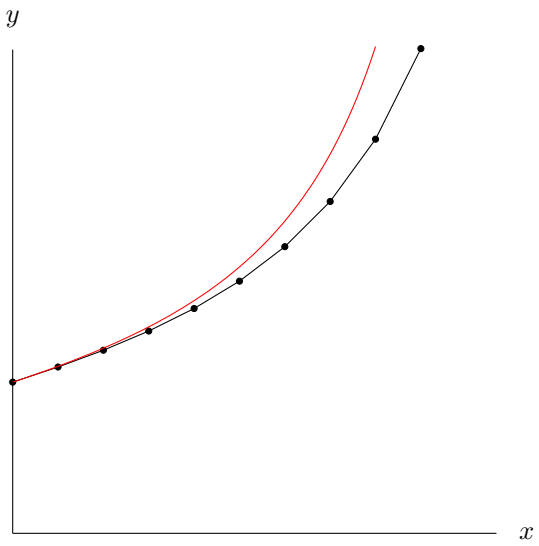
and here is the output of the program:

```
(0.1, 1.1)
(0.2, 1.211)
(0.3, 1.338)
(0.4, 1.487)
(0.5, 1.668)
(0.6, 1.896)
(0.7, 2.195)
(0.8, 2.607)
(0.9, 3.206)
```

We can visualize this by plotting points in the xy plane, and connecting them with line segments:



When we "connect the dots", we see what looks like a smooth curve, and you might think that what we're seeing is a very nice approximation to the graph of the solution $y(x)$. But observe what happens if we plot the approximations we made against the actual solution (shown in red):



As you can see, Euler's method consistently *underestimates* the solution - it's not the best.

Fortunately, computers never get tired these days, so you can make them do an essentially infinite amount of work. To produce the "actual solution", I just had my computer run Euler's method with 100 steps of size 0.01 - it took far less than a second.

If you check my work by having *your* computer run Euler's method with 1000 steps of size 0.001, you'll find that my solution is slightly incorrect, but the difference isn't easy to see from the graph.

In any case, the *idea* of Euler's method is that as we use smaller and smaller step sizes, the graphs that we draw by "connecting the dots" should become more and more accurate, and converge to the actual graph of the actual solution, in the limit as $h \to 0$.

To see this in action, consider the simplest possible initial value problem:

$$y' = y\ , \quad y(0) = 1$$

You should already know the solution:

$$y = e^x.$$

Let's try to rediscover this solution using Euler's method. In this case, the linear approximation formula is

$$y(x+h) \approx y(x) + hy'(x) = (1+h)y(x).$$

Repeatedly applying this approximation, we obtain:

$$\begin{aligned} y(0) &= 1 \\ y(h) &\approx (1+h)y(0) = (1+h) \\ y(2h) &\approx (1+h)y(h) = (1+h)^2 \\ y(3h) &\approx (1+h)y(2h) = (1+h)^3 \\ &\cdots \\ y(nh) &\approx (1+h)^n \end{aligned}$$

To see more clearly what's going on here, we can make the substitution $x = nh$, or equivalently $n = \frac{x}{h}$. This gives us the approximation

$$y(x) \approx (1+h)^{\frac{x}{h}} = \left((1+h)^{\frac{1}{h}}\right)^x.$$

Now, you may be aware that

$$\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e.$$

You may have even seen the number e *defined* using this limit. In any case, knowing that the value of the limit is e, we see that the actual solution of our equation is the limit of the approximate solutions:

$$y(x) = \lim_{h \to 0} \left((1+h)^{\frac{1}{h}} \right)^x = \left(\lim_{h \to 0} (1+h)^{\frac{1}{h}} \right)^x = e^x.$$

It can be proved mathematically that this always works, for any reasonable[1] first order differential equation: the approximations produced by Euler's method converge to the actual solution, in the limit as $h \to 0$.

It bears repeating that of all the methods you could use to solve differential equations numerically, Euler's method is probably one of the worst. If you want to actually learn how to do things like this properly, you'll need to read a book on numerical analysis.

[1] Specifically, the equation must satisfy the conditions stated in the existence and uniqueness theorem.

Series Solutions I. Another method for producing approximate solutions is called *repeated differentiation*. It relies on Taylor's formula: [2]

$$y(x) = \sum_{n=0}^{\infty} y^{(n)}(x_0) \frac{(x-x_0)^n}{n!}.$$

To use this formula to solve an initial value problem

$$y'(x) = f(x, y(x)) \ , \ \ y(x_0) = y_0$$

we can carry out a two step process:

(1) Repeatedly differentiate the equation, using the technique of "implicit differentiation".
(2) Substitute $x = x_0$ to obtain values for $y^{(n)}(x_0)$, and plug those values in to Taylor's formula.

As a simple example of this method, consider the differential equation

$$y' = 2y \ , \ \ y(3) = 4$$

To see clearly how the method works, let's make the dependence of y on x explicit in the equation:

$$y'(x) = 2y(x)$$

If we repeatedly differentiate the equation, we quickly observe a general pattern:

$$\begin{aligned} y'(x) &= 2y(x) \\ y''(x) &= 2y'(x) \\ y'''(x) &= 2y''(x) \\ &\cdots \\ y^{(n)}(x) &= 2y^{(n-1)}(x) \end{aligned}$$

Once we see this pattern, we can substitute the value $x = 3$ to obtain a general formula for $y^{(n)}(3)$. To get started, we use the given initial value $y(3) = 4$.

$$\begin{aligned} y(3) &= 4 \\ y'(3) &= 2y(3) = 2 \cdot 4 \\ y''(3) &= 2y'(3) = 2^2 \cdot 4 \\ y'''(3) &= 2y''(3) = 2^3 \cdot 4 \\ &\cdots \\ y^{(n)}(3) &= 2y^{(n-1)}(3) = 2^n \cdot 4 \end{aligned}$$

At each stage of the process, we made use of information we already knew. For example, we started with the initial condition $y(3) = 4$, and used it to determine $y'(3)$. Once we knew $y'(3)$ we could use its value to determine the value of $y''(3)$, and so on.

Once we have a formula for $y^{(n)}(3)$, we can substitute into Taylor's formula:

$$y(x) = \sum_{n=0}^{\infty} y^{(n)}(3) \cdot \frac{(x-3)^n}{n!} = \sum_{n=0}^{\infty} 2^n \cdot 4 \cdot \frac{(x-3)^n}{n!}$$

This gives an exact formula for the solution, which we can simplify using the basic Taylor series for e^x:

$$y(x) = 4 \sum_{n=0}^{\infty} \frac{(2 \cdot (x-3))^n}{n!} = 4 \sum_{n=0}^{\infty} \frac{(2x-6)^n}{n!} = 4e^{2x-6}.$$

Notice that all we needed to know here was the initial value $y(3)$. Once we knew that value, all of the remaining coefficients were uniquely determined!

[2] Recall from math 183 that Taylor's formula *is not valid for every single-variable function* $y(x)$. However, it is possible to prove the following mathematical theorem: if $y(x)$ is any solution of a differential equation $y' = f(x, y)$, and the multivariable Taylor formula for $f(x, y)$ is valid at any point (x_0, y_0) in the domain of $f(x, y)$, then the single variable Taylor formula for $y(x)$ is valid at the point $x = x_0$, at least within the radius of convergence of the Taylor series. So, under reasonable hypotheses on the differential equation we are trying to solve, everything in this section *can* be justified in a mathematically rigorous way.

As a second example, consider the initial value problem

$$y' = y^2 - x \ , \ \ y(0) = 1.$$

Let's apply the same method to this equation, but this time we'll suppress the dependence of y on x - this will make the algebra a bit less cluttered. We begin by repeatedly differentiating:

$$\begin{aligned} y' &= y^2 - x \\ y'' &= 2yy' - 1 \\ y''' &= 2y'y' + 2yy'' - 0 \\ y^{(4)} &= 2y''y' + 2y'y'' + 2y'y'' + 2yy''' = 6y''y' + 2yy''' \\ &\cdots \end{aligned}$$

In this case, no particular pattern jumps out at us. However, we can still solve for the first few values of $y^{(n)}(0)$, starting with the initial value $y(0) = 1$.

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= y(0)^2 - x = 1 - 0 = 1 \\ y''(0) &= 2y(0)y'(0) - 1 = 2 \cdot 1 \cdot 1 - 1 = 1 \\ y'''(0) &= 2y'(0)y'(0) + 2y(0)y''(0) - 0 = 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 = 4 \\ y^{(4)}(0) &= 6y''(0)y'(0) + 2y(0)y'''(0) = 6 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 4 = 14 \\ &\cdots \end{aligned}$$

Again, once we know the value of $y(0)$, each higher derivative is determined in a completely mechanical way! We can then apply Taylor's formula:

$$y(x) = 1 + 1 \cdot x + 1 \cdot \frac{x^2}{2!} + 4 \cdot \frac{x^3}{3!} + 14 \cdot \frac{x^4}{4!} + \cdots$$

In theory, nothing would prevent us from calculating *all* of the Taylor coefficients this way. In practice, we would wither and die before the task was complete, so we can only hope to obtain Taylor *approximations*.

Nevertheless, seeing how the process works, you can make two basic observations:

(1) An initial value problem of the form

$$y' = f(x, y) \ , \ \ y(x_0) = y_0$$

always has a solution.[3]

(2) While a solution $y(x)$ does always exist, it might only be defined on a small interval $(x_0 - \epsilon, x_0 + \epsilon)$, if the Taylor series does not have an infinite radius of convergence.[4]

These observations are a restatement of the existence theorem for first order equations (stated last week). To them we can add a third observation, based on the idea that we can solve for all of the derivatives $y^{(k)}(x_0)$ in terms of $y(x_0)$:

(3) An initial value problem of the form

$$y' = f(x, y) \ , \ \ y(x_0) = y_0$$

has *exactly one* solution. [5]

[3]Technically, we have only produced a Taylor series for a solution. It is not clear whether this Taylor series converges, and even if it did, it would not be clear that the function it converged to was actually a solution of the differential equation! These technicalities keep mathematicians up at night, but for most differential equations you will encounter, they don't matter: the solution will always be equal to its Taylor series, at any point where the series converges.

[4]Technically, the radius of convergence could end up being 0. For most differential equations you will encounter, the radius of convergence ends up being positive, but mathematicians do have the ability to cook up horrifying counterexamples.

[5]Technically, we have only seen that the *Taylor series* of the solution is determined by $y(x_0)$. In general it is possible for two infinitely differentiable functions to have the same Taylor series!

Mathematicians have verified this observation in a mathematical rigorous way, and obtained the following:

Existence and Uniqueness Theorem for Solutions of First Order Differential Equations. Suppose that the functions $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are both continuous in the vicinity of a point (x_0,y_0). Then the initial value problem

$$y' = f(x,y)\ ,\ \ y(x_0) = y_0.$$

has exactly one solution.

As an example of an equation to which this theorem *does not apply*, consider the initial value problem

$$y' = 2\sqrt{y}\ ,\ \ y(0) = 0.$$

This initial value problem has two different solutions,

$$y_1 = 0 \ \text{ and } \ y_2 = x^2.$$

However, the existence of more than one solution is actually consistent with the theorem, in this case! If we take the right hand side of the equation,

$$f(x,y) = 2\sqrt{y}$$

and differentiate it with respect to y,

$$\frac{\partial}{\partial y}\sqrt{y} = \frac{1}{2\sqrt{y}},$$

we get a function which is *discontinuous* at $y = 0$.

The same issue comes up in the repeated differentiation approach. Differentiating once, we obtain

$$y'' = \frac{y'}{\sqrt{y}},$$

which is a problem, because we now need to substitute $x = y = 0$, and this requires division by 0:

$$y''(0) = \frac{y'(0)}{\sqrt{0}}$$

So, the repeated differentiation method can't be directly applied, in this case.

However, it *can* be applied if we rearrange the equation slightly before differentiating it. Squaring both sides,

$$y'^2 = 4y$$

and *then* repeatedly differentiating, we obtain the following:

$$\begin{aligned} 2y'y'' &= 4y' \\ 2y''y'' + 2y'y''' &= 4y'' \\ &\cdots \end{aligned}$$

If we substitute $x = y = 0$, the original equation tells us that

$$y'(0)^2 = 4y(0) = 0.$$

This forces us to choose the value

$$y'(0) = 0.$$

The second equation

$$2y'(0)y''(0) = 4y'(0)$$

now says that $0 = 0$: this will hold no matter which value of $y''(0)$ we choose. Finally, the third equation,

$$2y''(0)^2 + 2y'(0)y'''(0) = 4y''(0),$$

simplifies to

$$2y''(0)^2 = 4y''(0),$$

and this has two solutions:

$$y''(0) = 0 \ \text{ or } \ y''(0) = 2.$$

If you follow these two cases to their logical conclusions, you will obtain the solutions $y_1 = 0$ and $y_2 = x^2$.

Beyond repeated differentiation, there is another method of finding series solutions (which is easier to apply in certain cases). The idea is to write the solution as a power series:

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

and substitute the series into the equation.

For example, suppose we want to solve the equation

$$y' = xy.$$

Then we can write y as a series,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

substitute this in both sides of our equation,

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots$$
$$xy = a_0x + a_1x^2 + a_2x^3 + \cdots + a_{n-1}x^n + \cdots$$

and set the two series equal to each other:

$$\begin{aligned} & a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots \\ = \quad & 0 \quad + a_0x \quad + a_1x^2 \quad + \cdots + a_{n-1}x^n + \cdots \end{aligned}$$

In order for two series to be equal, *each of their terms must individually be equal.* This is equivalent to Taylor's theorem, which gives a formula for the n^{th} term of any power series in terms of the n^{th} derivative of the function defined by that power series.

Notice that when we wrote out the two series, we put each power of x in the same column. This makes it easier to compare the corresponding coefficients, and is generally a good practice.

In our example, equating the coefficients gives the following result:

$$\begin{aligned} a_1 &= 0 \\ 2a_2 &= a_0 \\ 3a_3 &= a_1 \\ &\cdots \\ (n+1)a_{n+1} &= a_{n-1} \end{aligned}$$

After the first few terms, a general rule goes into effect, which allows us to calculate each coefficient in terms of the preceding ones:

$$a_{n+1} = \frac{a_{n-1}}{n+1}$$

A pattern like this is called a **recursion**. Be careful! The recursion may not apply for all values of n. In this case, we can only use the recursion for $n > 0$, because "a_{-1}" has no meaning. In general, you should always write out the first few terms by hand, so you can tell when your recursion goes into effect.

To see how to apply a recursion to quickly calculate coefficients, let's use it to find a_7:

$$a_7 = \frac{a_5}{7} = \frac{a_3}{7 \cdot 5} = \frac{a_1}{7 \cdot 5 \cdot 3} = 0.$$

More generally, we see that any *odd* coefficient will be zero:

$$a_{2k+1} = 0$$

Similarly, we can find a_8:

$$a_8 = \frac{a_6}{8} = \frac{a_4}{8 \cdot 6} = \frac{a_2}{8 \cdot 6 \cdot 4} = \frac{a_0}{8 \cdot 6 \cdot 4 \cdot 2}.$$

and in general, we see that the even terms will be given by the formula

$$a_{2k} = \frac{a_0}{2k(2k-2)(2k-4)\cdots 4 \cdot 2} = \frac{a_0}{2^k \cdot k(k-1)(k-2)\cdots 2 \cdot 1} = \frac{a_0}{2^k \cdot k!}.$$

Plugging in, we obtain the general solution of the equation:

$$y = \sum_{k=0}^{\infty} \frac{a_0}{2^k \cdot k!} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{2}\right)^k = a_0 e^{\frac{x^2}{2}}.$$

In general, the benefit of using this method is that having a recursion makes it easier to identify patterns and produce an explicit formula for the general solution in terms of known functions (if such a formula exists).

Both methods of finding series solutions (repeated differentiation and substitution) can also be applied to *second order* equations,

$$y'' = F(x, y, y').$$

For example, consider the initial value problem

$$y'' = 3y' - 2y \ , \ \ y(0) = 0 \ , \ \ y'(0) = 1$$

To solve this equation we can repeatedly differentiate:

$$\begin{aligned} y'' &= 3y' - 2y \\ y''' &= 3y'' - 2y' \\ y^{(4)} &= 3y''' - 2y'' \\ &\cdots \\ y^{(n)} &= 3y^{(n-1)} - 2y^{(n-2)} \end{aligned}$$

Setting $x = 0$, and applying the initial conditions, we obtain:

$$\begin{aligned} y''(0) &= 3y'(0) - 2y(0) \\ y'''(0) &= 3y''(0) - 2y'(0) \\ y^{(4)}(0) &= 3y'''(0) - 2y''(0) \\ &\cdots \\ y^{(n)}(0) &= 3y^{(n-1)}(0) - 2y^{(n-2)}(0) \end{aligned}$$

Observe that it is possible to solve for all of the derivatives $y^{(k)}(0)$ in terms of $y(0)$ and $y'(0)$! In general, repeated differentiation can always be used to find series solutions of initial value problems

$$y'' = F(x, y, y') \ , \ \ y(x_0) = y_0 \ , \ \ y'(x_0) = y_0',$$

because we will always be able to solve for $y^{(k)}(x_0)$ in terms of $y(x_0)$ and $y'(x_0)$.

A more mathematically precise version of this observation is:

Existence and Uniqueness Theorem for Solutions of Second Order Differential Equations. Suppose that the function

$$F(x, y, y')$$

and all of its partial derivatives are continuous in the vicinity of a point (x_0, y_0, y_0'). Then every initial value problem

$$y'' = F(x, y, y') \ , \ \ y(x_0) = y_0 \ , \ \ y'(x_0) = y_0'.$$

has exactly one solution, although it might only be defined on a small interval $(x_0 - \epsilon, x_0 + \epsilon)$.

Returning to the equation

$$y'' = 3y' - 2y,$$

suppose we wanted to apply the substitution method instead of repeatedly differentiating. Then we would write out y, y', and y'' as power series:

$$\begin{aligned} y &= c_0 + c_1 x + c_2 \frac{x^2}{2!} + c_3 \frac{x^3}{3!} + \cdots + \ c_n \frac{x^n}{n!} + \cdots \\ y' &= c_1 + c_2 x + c_3 \frac{x^2}{2!} + c_4 \frac{x^3}{3!} + \cdots + c_{n+1} \frac{x^n}{n!} + \cdots \\ y'' &= c_2 + c_3 x + c_4 \frac{x^2}{2!} + c_5 \frac{x^3}{3!} + \cdots + c_{n+2} \frac{x^n}{n!} + \cdots \end{aligned}$$

Notice that we have included a factorial in the denominator, and changed the letter we used for the coefficient. In terms of Taylor's formula, we have

$$c_n = y^{(n)}(0)$$

whereas before we would have had

$$a_n = \frac{y^{(n)}(0)}{n!}.$$

It is often better to include the factorials - this can make the recursion easier to solve.

In particular, using factorials is helpful for the equation we are currently trying to solve. When we substitute,

$$\begin{aligned} y'' &= c_2 + c_3 x + c_4 \frac{x^2}{2!} + c_5 \frac{x^3}{3!} + \cdots + c_{n+2} \frac{x^n}{n!} + \cdots \\ &= 3(c_1 + c_2 x + c_3 \frac{x^2}{2!} + c_4 \frac{x^3}{3!} + \cdots + c_{n+1} \frac{x^n}{n!} + \cdots) \\ &\quad -2(c_0 + c_1 x + c_2 \frac{x^2}{2!} + c_3 \frac{x^3}{3!} + \cdots + c_n \frac{x^n}{n!} + \cdots) \\ &= 3y' - 2y \end{aligned}$$

We obtain the recursion

$$c_{n+2} = 3c_{n+1} - 2c_n,$$

which is valid for $n \geq 0$.

Notice that this is identical to the recursion we obtained using repeated differentiation:

$$y^{(n)}(0) = 3y^{(n-1)}(0) - 2y^{(n-2)}(0),$$

which was valid for $n \geq 2$.

Now, you may not see how to solve for c_n based on this recursion. There's a clever trick for this, which uses matrices and eigenvalues! Notice that

$$\begin{bmatrix} c_{n+1} \\ c_n \end{bmatrix} = \begin{bmatrix} 3c_n - 2c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_n \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_{n-1} \\ c_{n-2} \end{bmatrix} = \cdots = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$$

The matrix power can be computed using eigenvectors and eigenvalues - try it!

For more on the substitution technique, see chapter 12 of your textbook. It's an important technique, which gets applied in crucial ways in quantum mechanics, and we'll see more examples of it in math 186.

Systems of First Order Equations. In many applications of differential equations, you will not be trying to keep track of a *single* quantity which is changing over time. Instead, there will be *many* different quantities, all of which are changing, and the rate of change of each quantity will be determined not only by *its* value, but by the values of the other quantities as well.

For example, in a complicated chemical reaction you might need to keep track of the concentrations of several different chemical compounds, and this will be controlled by several different reaction rates, all of which depend on the concentrations of all of the chemicals involved in the reaction.

In situations like this you will need to solve *systems* of first order equations,

$$\begin{aligned} \frac{dx_1}{dt} &= F_1(x_1, x_2, x_3, \dots) \\ \frac{dx_2}{dt} &= F_2(x_1, x_2, x_3, \dots) \\ \frac{dx_3}{dt} &= F_3(x_1, x_2, x_3, \dots) \\ &\dots \end{aligned}$$

where each $x_i = x_i(t)$ is an unknown function of t.

To visualize systems of first order equations, it is helpful to bring back an idea from 184: the concept of a *velocity field*. Recall that any vector field

$$\vec{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$$

can be thought of as representing the *velocity* of a moving fluid. In this context, the motion of any individual molecule in the fluid can be modeled as a parameterized curve $(x(t), y(t))$, whose velocity is at any point in time is equation to $\vec{F}(x(t), y(t))$. We can write this as follows:

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$$

$$\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}.$$

Separating out the components in this vector equation, we obtain a system of two first order equations:

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}$$

Any system of differential equations can be visualized using this framework, even if the equations have an unrelated physical meaning (concentrations of chemicals, populations of wolves and rabbits, etc.).

In this course we will only discuss systems of two differential equations with two unknown functions. One reason for this is that 2D systems are the easiest systems to visualize. But there is a deeper reason: systems of first order equations in 3 or more variables, or even non-autonomous systems in 2 variables, can exhibit *chaotic* behavior. Chaos can be very interesting, and in fact there is an entire discipline within mathematics devoted to dealing with it (*chaos theory*), but it's not something you'll want to get involved with just yet.

In general, it is also possible to consider systems where the velocity field $\vec{F} = \vec{F}(\vec{x}, t)$ is *time-dependent*. However, even time-dependent 2D systems can exhibit chaotic behavior. So, we will also restrict our attention to systems which are *autonomous* (time-independent).

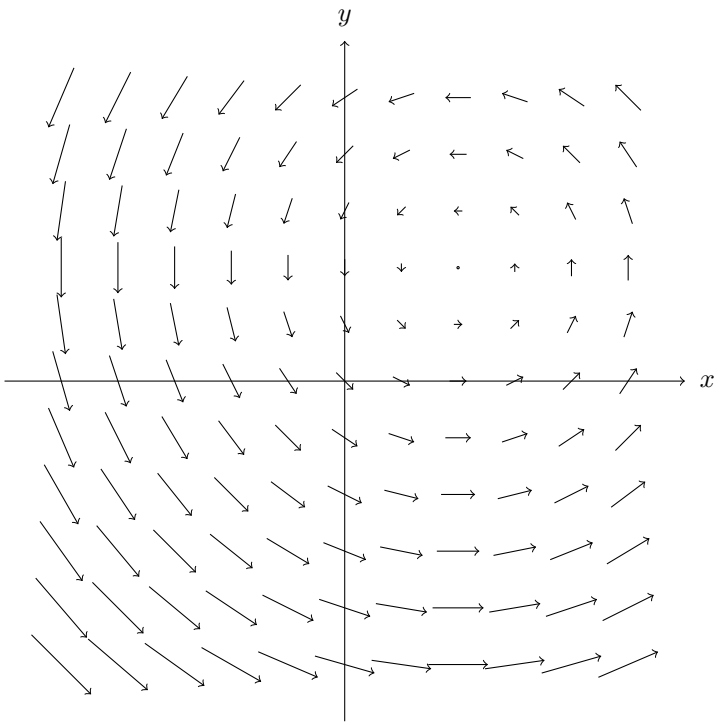
As a simple example of a 2D autonomous system, consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= 1 - y \\ \frac{dy}{dt} &= x - 1. \end{aligned}$$

The velocity field of this system is

$$\vec{F} = (1 - y)\hat{\imath} + (x - 1)\hat{\jmath}$$

and it looks like this:



Notice that we have been careful to draw all of our velocity vectors with the correct *magnitude* and *direction*. These aspects are very important - they reflect the *speed* of the solution, and its *direction of travel*.

Also notice that there is one point where we haven't drawn a velocity vector: the point $(1, 1)$. At this point, the velocity field vanishes:

$$\vec{F}(1,1) = (1-1)\hat{\imath} + (1-1)\hat{\jmath} = \vec{0}$$

A point like this is called an *equilibrium point*. In terms of the fluid flow analogy, an equilibrium point is a point where a molecule would be trapped and unable to move.[6]

Mathematically speaking, any equilibrium point gives us a *constant* solution of the system. For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a solution of the system above, because

$$\frac{d}{dt}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix}$$

We can visualize the non-constant solutions of the system by drawing *integral curves* - curves which are *everywhere tangent* to the velocity field. In the example above, the integral curves appear to be *circles*: [7]

[6] Here is a somewhat depressing example of an equilibrium point:

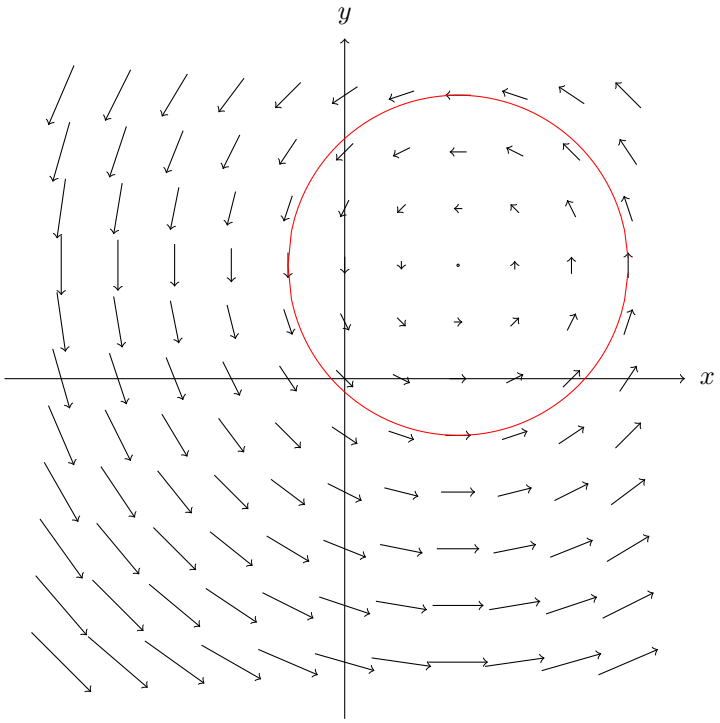
https://en.wikipedia.org/wiki/Great_Pacific_garbage_patch

In this case the velocity field is time-dependent due to temperature and pressure fluxuations, so the equilibrium moves around over time and the result is the "relatively stationary region" alluded to in the article.

[7] You can consider this a lucky guess. However, it can be justified using the multivariable chain rule, by observing that for a solution $(x(t), y(t))$ we would necessarily have

$$\frac{d}{dt}(x-1)^2 + (y-1)^2 = 2(x-1)\frac{dx}{dt} + 2(y-1)\frac{dy}{dt} = 2(x-1)(y-1) + 2(y-1)(1-x) = 0$$

and thefore $(x-1)^2 + (y-1)^2$ must be constant on any integral curve



But the integral curves alone don't tell us everything we need to know about the solutions $(x(t), y(t))$. In addition to knowing that the solutions move along circles, we want to know *how* they move along those circles - in which direction, and how fast.

To say it another way: the integral curves are determined by the *slope* of the velocity field. To obtain the solutions, we need to take into account the *magnitude* and *direction* of the velocity field.

In this particular example, we can notice that the integral curves are given by equations

$$(x-1)^2 + (y-1)^2 = R^2 \ , \quad R = \text{constant}.$$

Solving for y, under the assumption $y > 1$, we find that

$$y = 1 + \sqrt{R^2 - (x-1)^2}.$$

If we plug this back in to the first equation in our system,

$$\frac{dx}{dt} = 1 - y,$$

we obtain a differential equation for x:

$$\frac{dx}{dt} = -\sqrt{R^2 - (x-1)^2}$$

This can be solved by separation of variables - its solution is

$$x = 1 + R\cos t.$$

Substituting back into the equation for y, we find that

$$y = 1 + \sqrt{R^2 - (1 + R\cos t - 1)^2} = 1 + R\sqrt{1 - \cos^2 t} = 1 + R\sin(t).$$

This gives the solution of the system,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + R\cos t \\ 1 + R\sin t \end{bmatrix}.$$

To verify that this is a valid solution, notice that

$$\frac{d}{dt}\begin{bmatrix} 1 + R\cos t \\ 1 + R\sin t \end{bmatrix} = \begin{bmatrix} -R\sin t \\ R\cos t \end{bmatrix} = \begin{bmatrix} 1 - (1 + R\sin t) \\ (1 + R\cos t) - 1 \end{bmatrix}$$

Next week you will see a more systematic method of obtaining this solution, which applies to any *linear* system of differential equations.

In theory, it is possible to find solutions of any first order system, with arbitrary initial values $(x(0), y(0))$, even if the system is nonlinear. This result can be stated in a mathematically precise way, as follows:

Existence and Uniqueness Theorem for Systems of First Order Differential Equations. Suppose that the functions $P(x, y)$, $Q(x, y)$, and their partial derivatives are continuously differentiable in the vicinity of a point (x_0, y_0). Then there is a unique solution $(x(t), y(t))$ of the initial value problem

$$\frac{dx}{dt} = P(x, y) \ , \quad x(t_0) = x_0$$
$$\frac{dy}{dt} = Q(x, y) \ , \quad y(t_0) = y_0,$$

although it may not be possible to extend this solution beyond a small interval of the form $(t_0 - \epsilon, t_0 + \epsilon)$.

You may find it unsatisfying to see this result stated without any justification. If so, here is a detailed method which can be applied to construct solutions of arbitrary systems in two variables. Note that this method only works *in theory* - it is impossible to implement in most practical cases.

(1) Determine the *slope* of the velocity field, by dividing the two equations:

$$\begin{cases} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{cases} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{Q(x, y)}{P(x, y)}$$

The integral curves of the velocity field can then be obtained by solving the equation,

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)},$$

since their slope is equal to the slope of the velocity field at every point (by definition).

(2) Find the particular integral curve passing through (x_0, y_0), by solving the initial value problem

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \ , \quad y(x_0) = y_0.$$

This can be done as long as $P(x_0, y_0) \neq 0$, and results in an equation for the integral curve, in the form $y = f(x)$. If $P(x_0, y_0) = 0$ and $Q(x_0, y_0) \neq 0$, you can instead solve the initial value problem

$$\frac{dx}{dy} = \frac{P(x, y)}{Q(x, y)} \ , \quad x(y_0) = x_0,$$

obtaining an equation of the form $x = g(y)$. If $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$, then (x_0, y_0) is an equilibrium point, and the solution is constant: $(x, y) = (x_0, y_0)$.

(3) If the equation of the integral curve is $y = f(x)$, go back to the first equation of the original system and substitute for y:

$$\frac{dx}{dt} = P(x, f(x)) \ , \quad x(t_0) = x_0$$

The first of these equations can be used to solve for x as a function of t.

(4) Substitute $x(t)$ in the equation of the integral curve, to obtain y as a function of t:

$$y = f(x) = f(x(t))$$