Lecture 3: Probability and equal likelihood

Statistics 251

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Corrections to L2

Distributive Principle

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$$
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Poker hands example: note when counting hands we have already considered the overcount factor 5!.

Lecture Outline

Equal likelihood

Problems

Inclusion-Exclusion

More problems

Where are we?

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Events of equal likelihood

Suppose a sample space S has N events. If each event is equally likely, what is the probability of each event?

- If each event has probability p, then $p \cdot N = 1$, so $p = \frac{1}{N}$

What is $\mathbb{P}(A)$ for a general subset $A \subset S$?

- A consists of $|A|$ disjoint events, so the total probability is $\frac{|A|}{|S|}$.

Where are we?

Problems 1

What is the probability that the sum of two dice rolls is 3?

What is the probability that exactly 4 of 8 coin tosses are heads?

Problems 2

Roll 5 dice. What is the probability that exactly 2 of the dice show the number 1?

In a class of 60 students, what is the probability that none of them have student ID ending in 9?

Birthday paradox

In a room of N people, what is the probability that some two of them have the same birthday?

What is the smallest N for which this probability is above $\frac{1}{2}$?

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$2nd$

$$\frac{365}{366}\frac{364}{366}\cdots\frac{366-22}{366} = 0.4937$$

$$\frac{365}{366}\frac{364}{366}\cdots\frac{366-36}{366} = 0.1521$$

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Principle of inclusion-exclusion

Suppose we roll 2 dice and get two numbers a and b. What is the probability that either $a = b$ or a is even?

- Define $A = \{a = b\}$ and $B = \{a \text{ is even}\}$.
- $\mathbb{P}(A) = \frac{1}{6}$, $\mathbb{P}(B) = \frac{1}{2}$, and $\mathbb{P}(A \cap B) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Principle of inclusion-exclusion

More generally, suppose we have events $E_1, \ldots, E_n$. Then, we have

$$\begin{aligned}\mathbb{P}\Big(E_1 \cup \cdots \cup E_n\Big) = &\sum_{i=1}^{n} \mathbb{P}(E_i) - \sum_{i<j} \mathbb{P}(E_i \cap E_j) \\ &+ \sum_{i<j<k} \mathbb{P}(E_i \cap E_j \cap E_k) - \cdots \\ &+ (-1)^{n-1} P(\cap_{i=1}^{n} E_i).\end{aligned}$$

Derangements

Suppose a deck of n cards is shuffled. What is the probability that, after the shuffle, no card is in the same position?

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- Let E_i be the probability that card i is in position i after the shuffle.
- Because all orderings are equally likely, $\mathbb{P}(E_i) = \frac{1}{n}$.
- For any $i_1, \ldots, i_k$, $\mathbb{P}(E_{i_1} \cap \cdots \cap E_{i_k}) = \frac{(n-k)!}{n!}$.

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- In the principle of inclusion-exclusion, we have $\binom{n}{k}$ such terms, which means

$$\mathbb{P}\Big(E_1 \cup \cdots \cup E_n\Big) = \binom{n}{1} \cdot \frac{(n-1)!}{n!} - \binom{n}{2} \cdot \frac{(n-2)!}{n!} + \binom{n}{3} \cdot \frac{(n-3)!}{n!} - \cdots = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

- The answer is $1 - \mathbb{P}(E_1 \cup \cdots \cup E_n) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots = \frac{1}{e}$.

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Problems 3

What is the probability of getting a full house in poker?
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- Answer is $\frac{\#\{\text{unordered distinct 5-card sequences giving a full house}\}}{\#\{\text{unordered 5-card sequences}\}}$
- To get a full house, choose numbers and then suits: $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}$
- Total number: $\binom{52}{5}$, giving $\frac{6}{4165}$.

Problems 4

What is the probability of getting a two pair hand in poker?

[A two pair hand has 2 pairs of cards of the same rank and a 5th card of a different rank.]

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What is the probability of getting a two pair hand in poker?
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- Answer is $\frac{\#\{\text{unordered distinct 5-card sequences giving two pair}\}}{\#\{\text{unordered 5-card sequences}\}}$
- To get two pair, choose numbers and then suits:
  $\frac{13 \cdot 12}{2} \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{4}{1}$
- Total number: $\binom{52}{5}$, giving $\frac{198}{4165}$.