Lecture 2: Stochastic Process, Brownian Motion.

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Abstract

Summary of Stochastic process and Brownian motion

1 Stochastic Processes

Sequence of r.v.'s X_1, $X_2, \cdots, X_n, \cdots$ occuring at discrete times $t_1 < t_2 \cdots < t_n < \cdots$ is called a **discrete stochastic process**, with joint distribution $F_{X_{i_1}, X_{i_2}, \cdots, X_{i_j}}$, $i_j = 1, 2, \cdots$ as its probability law.

Continuous Stochastic Process: $X(t) = X(t, \omega)$, $t \in [0,1]$ or $[0, \infty)$, over probability space (Ω, A, P), is a function of two variables, $X : [0,1] \times \Omega \to R$, where X is a r.v. for each t, for each ω, we have a continuous sample path (realization/trajectory/configuration) of the process.

• Quantities on time variability: $\mu(t) = E(X(t))$, $\sigma^2(t) = Var(X(t))$, covariance:

$$C(s,t) = E((X(s) - \mu(s))(X(t) - \mu(t))),$$

for $s \neq t$.

Process with independent increment: $X(t_{j+1}) - X(t_j)$, $j = 0, 1, 2, \cdots$ are independent

Gaussian Process: all joint distributions are Gaussian.

Standard Wiener Process (Brownian Motion): Gaussian process $W(t)$, $t \geq 0$, with independent increment, and:

$$W(0) = 0 \; w.p.1, \; E(W(t)) = 0, \; Var(W(t) - W(s)) = t - s,$$

for all $s \in [0, t]$.

B.M. Covariance: $C(s,t) = \min(s,t)$.

Stationary Process: all joint distributions are translation (along time) invariant.

Ornstein-Uhlenbeck Process: Gaussian process with $X(0)$ unit Gaussian, $E(X(t)) = 0$, covariance $E(X_s X_t) = e^{-\gamma|t-s|}$ for $s, t \in R$, $\gamma > 0$.

• Note: B.M. Covariance: $C(s,t) = \min(s,t)$, not stationary. O-U stationary.

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2 Diffusion Process

Suppose joint distribution of $X(t)$ has density $p(t_1, x_1; t_2, x_2; \cdots; t_k, x_k)$, define conditional probability:

$$P(X(t_{n+1}) \in B | X(t_i) = x_i, i = 1:n) = \frac{\int_B p(t_1, x_1, \cdots, t_n, x_n; t_{n+1}, y)\, dy}{\int p(t_1, x_1, \cdots, t_n, x_n; t_{n+1}, y)\, dy}$$

for B any Borel set of R.

- *Markov Process* if:

$$P(X(t_{n+1}) \in B | X(t_i) = x_i, i = 1:n) = P(X(t_{n+1}) \in B | X(t_n) = x_n).$$

It means transition probability:

$$P(t_1, x_1, \cdots, t_{n-1}, x_{n-1}, \mathbf{s}, \mathbf{x}; t, B) = P(\mathbf{s}, \mathbf{x}; t, B) = \int_B p(\mathbf{s}, \mathbf{x}; t, y)\, dy.$$

- *Chapman-Kolmogorov (C-K) equation*:

$$p(s, x; t, y) = \int_{R^1} p(s, x; \tau, z)\, p(\tau, z; t, y)\, dz,$$

for $s \le \tau \le t$.

Eg1. Wiener process: $dX = dW$

$$p(s, x; t, y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\{-\frac{(y-x)^2}{2(t-s)}\},$$

Eg2. O-U: $dX = -Xdt + dW$ ($\gamma = 1$, proof can be found later in section 3)

$$p(s, x; t, y) = \frac{1}{\sqrt{2\pi(1 - e^{-2(t-s)})}} \exp\{-\frac{(y - xe^{-(t-s)})^2}{2(1 - e^{-2(t-s)})}\},$$

Diffusion Process: Markov process with transition density is called *diffusion process* if the following limits exist ($\forall \epsilon$):
Jump:

$$\lim_{t \to s^+} \frac{1}{t-s} \int_{|y-x| > \epsilon} p(s, x; t, y) dy = 0,$$

Drift:

$$\lim_{t \to s^+} \frac{1}{t-s} \int_{|y-x| \le \epsilon} (y-x) p(s, x; t, y) dy = a(s, x),$$

Diffusion:

$$\lim_{t \to s^+} \frac{1}{t-s} \int_{|y-x| \le \epsilon} (y-x)^2 \, p(s,x;t,y) dy = b^2(s,x).$$

• Note: diffusion process and Markov process defined in distribution sense, continuous process is defined in strong sense. If a Markov process is defined by a continuous process then it does not jump.

By analytic derivation, we can calculate them by:

$$a(s,x) = \lim_{t \to s^+} \frac{1}{t-s} E(X(t) - X(s) | X(s) = x),$$

$$b^2(s,x) = \lim_{t \to s^+} \frac{1}{t-s} E((X(t) - X(s))^2 | X(s) = x).$$

a: drift coefficient, b: diffusion coefficient.
Wiener: $(a,b) = (0,1)$, O-U: $(a,b) = (-\gamma\, x, \sqrt{2\gamma})$. (Calculation in next part)

• If a and b are moderately regular functions, $p(s,x;t,y)$ satisfies **Kolmogorov equation**:
Forward:

$$p_t + (a(t,y)p)_y = \frac{1}{2}(b^2(t,y)p)_{yy},$$

Backward:

$$p_s + a(s,x)p_x = -\frac{1}{2}b^2(s,x)p_{xx}.$$

Forward equation is also called Fokker-Planck equation.

Eg1. The transition density of Wiener obeys equations:

$$p_t = \frac{1}{2}p_{yy}, \ (s,x) \text{ fixed}$$

forward equation ($t > s$),

$$p_s = -\frac{1}{2}p_{xx}, \ (t,y) \text{ fixed}$$

backward equation ($s < t$).

3 Calculating Drift and Diffusion

Eg1. Brownian motion, using independence of increment, we see that:

$$E(X(t) - X(s) | X(s) = x) = 0,$$

so

$$a(s,x) = \lim_{t \to s^+} \frac{1}{t-s} E(X(t) - X(s) | X(s) = x) = 0;$$

$$E((X(t)-X(s))^2|X(s)=x)=t-s,$$

$$b^2(s,x)=\lim_{t\to s^+}\frac{1}{t-s}E((X(t)-X(s))^2|X(s)=x)=1.$$

Eg2. O-U process:

We start from the fact: $\tau\geq 0$, $X(t+\tau)-e^{-\gamma t}X(\tau)$ is independent of $(\omega : X(s), s\leq\tau)$. X and $X(t+\tau)-e^{-\gamma t}X(\tau)$ are both Gaussian. Covariance ($s\leq\tau$):

$$\begin{aligned} E[(X(t+\tau)-e^{-\gamma t}X(\tau))X(s)] &= C(s,t+\tau)-e^{-\gamma t}C(s,\tau) \\ &= e^{-\gamma|t+\tau-s|}-e^{-\gamma t-\gamma|\tau-s|} \\ &= 0. \end{aligned} \tag{3.1}$$

To find transition probability:

$$\begin{aligned} & P(X(t+\tau)\in A|X(\tau)=x)= \\ & P(X(t+\tau)-e^{-\gamma t}X(\tau)\in A-e^{-\gamma t}X(\tau)|X(\tau)=x) \\ = \; & P(X(t+\tau)-e^{-\gamma t}X(\tau)\in A-e^{-\gamma t}x). \end{aligned} \tag{3.2}$$

So R.V. $X(t+\tau)-e^{-\gamma t}X(\tau)$ is mean zero, Gaussian, and it variance can be calculated:

$$\begin{aligned} & E[(X(t+\tau)-e^{-\gamma t}X(\tau))^2]= \\ & E[(X(t+\tau)-e^{-\gamma t}X(\tau))X(t+\tau)] \\ & =1-e^{-2\gamma t}. \end{aligned} \tag{3.3}$$

(3.2) and (3.3) imply:

$$p(s,x;t,y)=\frac{1}{\sqrt{2\pi(1-e^{-2\gamma(t-s)})}}\exp\{-\frac{(y-xe^{-\gamma(t-s)})^2}{2(1-e^{-2\gamma(t-s)})}\}.$$

Calculate drift:

$$\begin{aligned} & E(X(t)-X(s)|X(s)=x)=E[X(t)-e^{-\gamma|t-s|}X(s) \\ & +e^{-\gamma|t-s|}X(s)-X(s)|X(s)=x] \\ & =(e^{-\gamma|t-s|}-1)x, \end{aligned} \tag{3.4}$$

$$a(s,x)=-\gamma x.$$

Calculate diffusion:

$$\begin{aligned} & E((X(t)-X(s))^2|X(s)=x)=E[(X(t)-e^{-\gamma|t-s|}X(s)+(e^{-\gamma(t-s)}-1)x)^2] \\ & =1-e^{-2\gamma(t-s)}+(e^{-\gamma(t-s)}-1)^2x^2, \end{aligned} \tag{3.5}$$

Note $1 - e^{-2\gamma(t-s)} \approx 2\gamma(t-s)$ while, $(e^{-\gamma(t-s)} - 1)^2 \approx \gamma^2(t-s)^2$, so

$$b^2(s,x) = \lim_{t \to s^+} \frac{1}{t-s} E((X(t) - X(s))^2 | X(s) = x) = 2\gamma.$$

Over small time interval $[s,t]$, using drift-diffusion information, we see that O-U is related to BM as (to leading order):

$$X(t) - X(s) = -\gamma X(s)(t-s) + \sqrt{2\gamma}(W(t) - W(s)),$$

where $W(t)$ denotes BM; or in differential form:

$$dX = -\gamma X dt + \sqrt{2\gamma} dW,$$

The term $-\gamma X dt$ physically means damping.

4 From Random Walk to Brownian Motion

Divide time interval $[0,1]$ into N equal length subintervals $[t_i, t_{i+1}]$, $i = 0, 1, \cdots, N$. Consider a walker making steps $\pm\sqrt{\delta t}$, $\delta t = 1/N$ with probability 1/2 each, starting from $x = 0$. In n steps, the walker's location is:

$$S_N(t_n) = \sqrt{\delta t} \sum_{i=1}^{n} X_i, \tag{4.6}$$

where X_i are independent two point r.v's taking ± 1 with equal probability. Define a piecewise continuous function:

$$S_N(t) = S_N(t_n),\ t \in [t_n, t_{n+1}],\ n \leq N - 1.$$

S_N has independent increment $X_1\sqrt{\delta t}$, $X_2\sqrt{\delta t}$ etc for given subintervals, and in the limit $N \to \infty$ tends to a process with independent increment. Moreover:

$$E(S_N) = 0,\ Var(S_N(t)) = \delta t \left[\frac{t}{\delta t}\right].$$

In the limit $N \to \infty$, $Var(S_N(t)) \to t$. Applying Central Limit Theorem,$\forall t$, the approximate process $S_N(t)$ converges in law to a process with independent increment, zero mean, variance t, and Gaussian. So it is a BM.

Remark: Now you should see why in the ant problem, we do not scale the step linearly!

Using $S_N(t)$ is a way to *numerically construct BM* as well. The X_i's are generated from $U(0,1)$ as: $X_i = 1$ if $U \in [0, 1/2]$; $X_i = -1$, if $U \in (1/2, 1]$.

Try the 2 line Matlab code to generate a BM sample path:

```
rand('state',0); N=1e4; dt=1/N;
w=sqrt(dt)*cumsum([0;rand(N,1)]); plot([0:dt:1],w);
```

An alternative way is to replace two point X_i's by i.i.d unit Gaussian r.v's. The code becomes:

```
randn('state',0); N=1e4; dt=1/N;
w=sqrt(dt)*cumsum([0;randn(N,1)]); plot([0:dt:1],w);
```

cumsum is a fast summation on vector input. Change the state number from 0 to 10 (or a larger number if you are having fun !) to see different sample paths (see Figure 1).

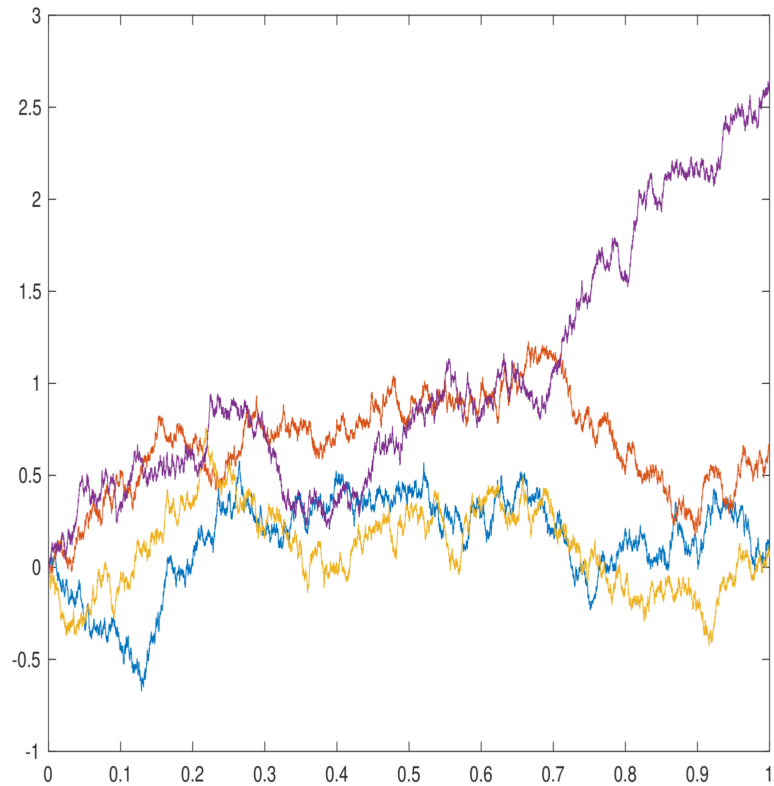

Figure 1: Four Sample Paths of Numerical Approximation of Brownian Motion on [0,1].

BM sample path is almost surely continuous. Kolmogorov criterion:

$$E(|X(t) - X(s)|^a) \leq C|t - s|^{1+b},$$

for a, b, C positive. For BM, $a = 4$, $b = 1$, $C = 3$.

Kolmogorov Continuity Theorem:

Let (S, d) be some complete metric space, and let $X : [0, +\infty) \times \Omega \to S$ be a stochastic process. Suppose that for all times $T > 0$, there exist positive constants α, β, K such that $\mathbb{E}\left[d\left(X_t, X_s\right)^\alpha\right] \leq K|t - s|^{1+\beta}$ for all $0 \leq s, t \leq T$. Then there exists a modification $\tilde{X}$ of X that is a continuous process, i.e. there exists a process $\tilde{X} : [0, +\infty) \times \Omega \to S$ such that

- $\tilde{X}$ is sample-continuous;
- for every time $t \geq 0, \mathbb{P}\left(X_t = \tilde{X}_t\right) = 1$ (modification)

Furthermore, the paths of $\tilde{X}$ are locally γ -Hölder-continuous for every $0 < \gamma < \frac{\beta}{\alpha}$.

5 BM via Random Fourier Series

Consider defining

$$Z(t) = \sum_{k=1}^{\infty} \varphi_k(t) Y_k,$$

by Y_k coefficient $\varphi_k(t)$ basis to be decided. For simplicity we assume Y_k i.i.d.

Now we assign some normalization on closed interval I,

$$\sum_{k=1}^{\infty} |\varphi_k(t)|^2 < \infty,\ t \in I.$$

$$\sum_{k=1}^{\infty} E|\varphi_k(t) Y_k|^2 = \sum_{k=1}^{\infty} |\varphi_k(t)|^2 < \infty,$$

the last equality needs $EY_i^2 = 1$. In this way the summing sequence converges in L_2 to $Z(t)$, $\forall t \in I$.

Let $EY_k = 0$ then $E[Z(t)] = 0$. The covariance is

$$C(t, s) = \sum_{k=1}^{\infty} \varphi_k(t)\, \varphi_k(s),$$

To match Z with BM, require:

$$\min(t, s) = \sum_{k=1}^{\infty} \varphi_k(t)\, \varphi_k(s).$$

Let $I = [0, \pi]$, fact:

$$\min(t, s) = \frac{ts}{\pi} + \frac{2}{\pi} \sum_{k \geq 1} \frac{\sin kt \sin ks}{k^2}.$$

So consider taking Y_i, $i = 1, 2, \cdots$ be $N(0, 1)$

$$W(t) = BM = \frac{t}{\sqrt{\pi}} Y_0 + \sqrt{\frac{2}{\pi}} \sum_{k \geq 1} \frac{\sin kt}{k} Y_k, \tag{5.7}$$

$t \in [0, \pi]$, $W(t)$ standard BM. By truncating the random Fourier series, we have a second way to generate BM.

Remark: Fourier construction makes an L_2 approximation to all BM path with finite number of random variable that is easy to generate. It is the starting point of a method called Wiener Chaos Expansion (PCE, gPC, etc..) which applies in a field called Uncertainty Quantification. The solution of stochastic partial differential equation like $u_t = Lu + dW_t$ is represented by orthogonal polynomials of Y_k.

6 Spectral Representation of stationary process

Consider stationary process, e.g. O-U. Covariance $C(t, s) = C(t - s)$, $C(\cdot)$ even function, and: $\forall \{a_i\} \subset R$,

$$\sum_{k,j} a_k a_j C(t_k - t_j) = E(|\sum_k a_k X(t_k)|^2) \geq 0,$$

$C(\cdot)$ is nonnegative definite and symmetric. Bochner theorem:

$$C(t) = \int_{-\infty}^{\infty} e^{2\pi i t \lambda}\, dF(\lambda), \tag{6.8}$$

R^1 Function $F(\lambda)$ is nondecreasing, right continuous, $F(+\infty) - F(-\infty) = C(0)$. We call F *spectral distribution function* of process $X(t)$
Assuming some regularity, $F'(\lambda)$ *spectral density* can be derived by Fourier transform,

$$F'(\lambda) = \int_{R^1} C(s) e^{-2\pi i \lambda s}\, ds = \int_{R^1} C(s)\, \cos(2\pi\lambda s)\, ds. \tag{6.9}$$

Just like finding a random Fourier series for BM from its covariance, one can construct a random Fourier integral for $X(t)$:
Let $Z(\lambda)$ be a process with orthogonal increments:

$$E[(Z(a) - Z(b))(Z(a') - Z(b'))] = 0,$$

if $(a, b) \cap (a', b')$ empty, and

$$E[(Z(a) - Z(b))^2] = F(a) - F(b),$$

Then

$$\hat{X}(t) = \int_{-\infty}^{\infty} e^{2\pi i t \lambda}\, dZ(\lambda). \tag{6.10}$$

has the same distribution with X.

The random integral $\int g(\lambda)\, dZ(\lambda)$ is defined as L_2 limit of finite Stieltjes sum:

$$\sum g_k[Z(\lambda_k) - Z(\lambda_{k-1})],$$

if $g \in L^2(dF)$.
Examples of Spectral Densities:
(1) O-U: $C(s) = e^{-\gamma|s|}$, taking Fourier transform (6.9):

$$F'(\lambda) = \frac{2\gamma}{\gamma^2 + 4\pi^2\lambda^2}.$$

(2) Gaussian white noise: $C(s) = \delta(s)$. $F'(\lambda) = 1$. The discrete Stieltjes integral does not converges!
Alternatively, we approximate it by:

$$X_h(t) = (W(t+h) - W(t))/h,$$

small $h > 0$. Process X^h has covariance and spectral density:

$$C_h(s,t) = \frac{1}{h}\max(0, 1 - |t-s|/h),$$

$$F_h'(\lambda) = \sin^2(2\pi\lambda h)/(\pi\lambda h)^2,$$

broad band spectrum, X_h called colored noise. In the limit $h \to 0$, C_h converges to delta function, X_h converges in some weak sense to white noise. ('derivative' of BM)