Lecture 15: Other continuous distributions and change of variables (univariate)

Statistics 251

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Lecture Outline

Gamma Distribution

Cauchy Distribution

Change of variable

Where are we?

Definition

A random variable is said to have a gamma distribution with parameters (α, λ), $\lambda > 0$, $\alpha > 0$, if its density function is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\Gamma(\alpha)$, called the gamma function, is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$$

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Integration by parts!

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$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty e^{-y} y^{\alpha-1} dy \\ &= -e^{-y} y^{-\alpha-1}|_0^\infty + \int_0^\infty e^{-y}(\alpha-1) y^{\alpha-2} dy \\ &= \end{aligned}$$

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So,

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) \\ &= \cdots \\ &= (n-1)(n-2)\cdots 3 \cdot 2\Gamma(1)\end{aligned}$$

Note $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$,
So $\Gamma(n) =$

Propositions

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If X follows a gamma distribution with parameter (α, λ), then $\mathbb{E}[X] = \frac{\alpha}{\lambda}$.

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What about $\mathrm{Var}(X)$?

Where are we?

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Cauchy Distribution

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If X follows a Cauchy distribution with parameter 0, $\mathbb{E}X$ is undefined!

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Theorem

Let X be a continuous random variable having probability density function f_X. Suppose that $g(x)$ is a strictly monotonic (increasing or decreasing), differentiable (and thus continuous) function of x. Then the random variable Y defined by $Y = g(X)$ has a probability density function given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

where $g^{-1}(y)$ is defined to equal that value of x such that $g(x) = y$.

Example: $Y = X^n$

Let X be a continuous nonnegative random variable with density function f, and let $Y = X^n$. Find f_Y, the probability density function of Y.

Example: Gamma Distribution

If X follows a gamma distribution with parameter (α, λ), then cX ($c > 0$) follows a Gamma distribution with parameter $(\alpha, \lambda/c)$