Lecture 4: Conditional probability

Statistics 251

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Lecture Outline

Definition of conditional probability

Problems

Multiplication rule

Where are we?

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Definition of conditional probability

Let S be a sample space and $F \subset S$ a subset (event).

- Suppose a random sample in S is drawn.
- The **conditional probability** of another event E *given* F is the probability that E happened given that F happened. Quantitatively, it is

$$\mathbb{P}(E \mid F) := \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

This is known as "the probability of E conditioned on F".

Definition of conditional probability

The conditional probability $\mathbb{P}(- \mid F)$ satisfies axioms of a probability space:

- $0 \leq \mathbb{P}(E \mid F) \leq 1$
- $\mathbb{P}(S \mid F) = 1$
- $\mathbb{P}\Big(\bigcup_{i \in I} E_i \mid F\Big) = \sum_i \mathbb{P}(E_i \mid F)$ for E_i disjoint and I countable.

Intepretation of $\mathbb{P}(- \mid F)$:

- Probabilities of events outside F are set to 0;
- Probabilities of events inside F are multiplied by $\frac{1}{\mathbb{P}(F)}$.

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Problem 1

Graph the probability that American Airlines will go bankrupt, given a sequence of events.

- Flights between US and China were canceled
- Europe travel ban
- Federal airline bailout
- ...

Problem 2

A coin is flipped twice. Assuming that all four points in the sample space $S = \{(h, h), (h, t), (t, h), (t, t)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that
(a) the first flip lands on heads?

(b) at least one flip lands on heads

Problem 3

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- Sample space is $S = \{\text{disease}, \text{no disease}\} \times \{\text{positive}, \text{negative}\}$
- $\mathbb{P}(\text{positive}) = 0.9 \cdot p + 0.1 \cdot (1 - p) = 0.1 + 0.8 \cdot p$
- $\mathbb{P}(\text{disease}, \text{positive}) = 0.9 \cdot p$
- $\mathbb{P}(\text{disease} \mid \text{positive}) = \frac{0.9 \cdot p}{0.9 \cdot p + 0.1 \cdot (1 - p)}$
- If p small, $\approx 9p$. If p large, ≈ 1.

Monty Hall problem

A TV show in the 1970s had the following game:

- There is a prize behind 1 of 3 doors, all equally likely.
- You point to a door, say A. The host opens one of the other two doors, say B, and shows you it does not have a prize.
- You get to open a door and claim what is behind it. Should you stay with A or switch to C?

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$$\frac{\mathbb{P}(\text{prize at A} \cap \text{you chose A and host points at B})}{\mathbb{P}(\text{you chose A and host points at B})} = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 0+\frac{1}{3}\cdot 1}$$
$$= \frac{1}{3}.$$

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For events $E_1, \ldots, E_n$, we have

$$\mathbb{P}\Big(E_1 \cap \cdots \cap E_n\Big) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2 \mid E_1) \cdots \mathbb{P}(E_n \mid E_1 \cap E_2 \cap \cdots \cap E_{n-1}).$$

To have $E_1 \cap \cdots \cap E_n$, must first have E_1, then E_2 given E_1, then E_3 given $E_1 \cap E_2$, then E_4 given $E_1 \cap E_2 \cap E_3$, ...

Example 1

Let E_i be the probability that a roll of a die lies outside $\{1, \ldots, i\}$. Then $\mathbb{P}(E_i) = \frac{6-i}{6}$.

- What is $\mathbb{P}(E_4 \mid E_1 \cap E_2 \cap E_3)$?

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- What is $\mathbb{P}(E_4 \mid E_1 \cap E_2 \cap E_3)$?
- The conditional sample space contains $\{4, 5, 6\}$, and $E_1 \cap E_2 \cap E_3 \cap E_4 = \{5, 6\}$, so the conditional probability is

$$\frac{|\{5, 6\}|}{|\{4, 5, 6\}|} = \frac{2/6}{3/6}.$$

Example 2

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Define events E_i, $i = 1, 2, 3, 4$, as follows:

$E_1 =${the ace of spades is in any one of the piles}

$E_2 =${the ace of spades and the ace of hearts are in different piles}

$E_3 =${the aces of spades, hearts, and diamonds are all in different piles}

$E_4 =${all 4 aces are in different piles}