Lecture 16: General Weak Approximation

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Abstract

Introducing weak schemes based on Ito-Taylor expansion and the convergence theorem.

Recall general rule of convergence, a weak order $\beta = 1, 2, 3, \cdots$ scheme needs all of the multiple Ito integrals from the Ito-Taylor expansion in the set $\Gamma_\beta = \{\alpha : l(\alpha) \le \beta\}$. Here l is the length of the index α. Note that is different from the strong scheme index set A_γ which also depends on the number of zeros in the index $n(\alpha)$.

1 Explicit Weak RK Schemes

1.1 Order 2 Schemes

Again we start with $d = m = 1$,

$$\begin{aligned} Y_{n+1} =& Y_n + a\Delta + b\Delta W \\ & + L^0 a I_{(0,0)} + L^1 a I_{(1,0)} + L^0 b I_{(0,1)} + L^1 b I_{(1,1)} \\ =& Y_n + a\Delta + b\Delta W \\ & + L^0 a \frac{\Delta^2}{2} + L^1 a \Delta Z + L^0 b(\Delta W \Delta - \Delta Z) + L^1 b \frac{(\Delta W)^2 - \Delta}{2} \end{aligned}$$

In deriving Taylor weak schemes, we also replace ΔW by $\Delta \hat{W}$, ΔZ by $\frac{1}{2}\Delta\hat{W}\Delta$ where one may choose $\hat{W}$ as $N(0, \Delta)$, or 3 -point random variable taking $\pm\sqrt{3\Delta}$ with prob 1/6 each, and zero with prob 2/3. So,

$$Y_{n+1} = Y_n + a\Delta + b\Delta\hat{W} + L^0 a \frac{\Delta^2}{2} + (L^1 a + L^0 b)\frac{\Delta\hat{W}\Delta}{2} + L^1 b \frac{(\Delta\hat{W})^2 - \Delta}{2}$$

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Now a step further, consider supporting values

$$\bar{\Upsilon} = Y_n + a\Delta + b\Delta\hat{W}$$
$$\bar{\Upsilon}^{\pm} = Y_n + a\Delta \pm b\sqrt{\Delta},$$

then Platen, in the autonomous case $d = 1, 2, \ldots$ with scalar noise $m = 1$, the following explicit order 2.0 weak scheme:

$$\begin{aligned} Y_{n+1} = Y_n &+ \frac{1}{2}(a(\bar{\Upsilon}) + a)\Delta \\ &+\frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) + b\left(\bar{\Upsilon}^{-}\right) + 2b\right)\Delta\hat{W} \\ &+ \frac{1}{4}\left(b\left(\bar{\Upsilon}^{+}\right) - b\left(\bar{\Upsilon}^{-}\right)\right)\left\{(\Delta\hat{W})^2 - \Delta\right\}\Delta^{-1/2}. \end{aligned}$$

For multi-dimensional case,

$$\begin{aligned} &Y_{n+1} = Y_n + \tfrac{1}{2}(a(\bar{\Upsilon}) + a)\Delta \\ &+\tfrac{1}{4}\textstyle\sum_{j=1}^{m}\Big[\left(b^j\left(\bar{R}_{+}^{j}\right) + b^j\left(\bar{R}_{-}^{j}\right) + 2b^j\right)\Delta\hat{W}^j \\ &\quad + \textstyle\sum_{\substack{r=1 \\ r\neq j}}^{m}\left(b^j\left(\bar{U}_{+}^{r}\right) + b^j\left(\bar{U}_{-}^{r}\right) - 2b^j\right)\Delta\hat{W}^j\Big] \\ &+\tfrac{1}{4}\textstyle\sum_{j=1}^{m}\Big[\left(b^j\left(\bar{R}_{+}^{j}\right) - b^j\left(\bar{R}_{-}^{j}\right)\right)\left\{\left(\Delta\hat{W}^j\right)^2 - \Delta\right\} \\ &\quad + \textstyle\sum_{\substack{r=1 \\ r\neq j}}^{m}\left(b^j\left(\bar{U}_{+}^{r}\right) - b^j\left(\bar{U}_{-}^{r}\right)\right)\left\{\Delta\hat{W}^j\Delta\hat{W}^r + V_{r,j}\right\}\Big]\Delta^{-1/2} \end{aligned}$$

with supporting values

$$\bar{\Upsilon} = Y_n + a\Delta + \sum_{j=1}^{m} b^j\Delta\hat{W}^j, \quad \bar{R}_{\pm}^{j} = Y_n + a\Delta \pm b^j\sqrt{\Delta}$$

and

$$\bar{U}_{\pm}^{j} = Y_n \pm b^j\sqrt{\Delta}$$

Here the $\Delta\hat{W}^j$ for $j = 1, 2, \ldots, m$ are independent random variables either 3-point or normal and the V_{j_1,j_2} are independent two-point distributed random variables with

$$P\left(V_{j_1,j_2} = \pm\Delta\right) = \frac{1}{2}$$

for $j_2 = 1, \ldots, j_1 - 1$,

$$V_{j_1,j_1} = -\Delta$$

and

$$V_{j_1,j_2} = -V_{j_2,j_1}.$$

1.2 Order 3 schemes for scalar additive noise

In the autonomous case $d = 1, 2, \ldots$ with $m = 1$ we have in vector form the explicit order 3.0 weak scheme for scalar additive noise

$$
\begin{aligned}
Y_{n+1} = Y_n &+ a\Delta + b\Delta\hat{W} \\
&+ \frac{1}{2}\left(a_\zeta^+ + a_\zeta^- - \frac{3}{2}a - \frac{1}{4}\left(\tilde{a}_\zeta^+ + \tilde{a}_\zeta^-\right)\right)\Delta \\
&+ \sqrt{\frac{2}{\Delta}}\left(\frac{1}{\sqrt{2}}\left(a_\zeta^+ - a_\zeta^-\right) - \frac{1}{4}\left(\tilde{a}_\zeta^+ - \tilde{a}_\zeta^-\right)\right)\zeta\Delta\hat{Z} \\
&+ \frac{1}{6}\left[a\left(Y_n + \left(a + a_\zeta^+\right)\Delta + (\zeta+\rho)b\sqrt{\Delta}\right) - a_\zeta^+ - a_\rho^+ + a\right] \\
&\times \left[(\zeta+\rho)\Delta\hat{W}\sqrt{\Delta} + \Delta + \zeta\rho\left\{(\Delta\hat{W})^2 - \Delta\right\}\right]
\end{aligned}
$$

with

$$
a_\phi^\pm = a\left(Y_n + a\Delta \pm b\sqrt{\Delta}\phi\right)
$$

and

$$
\tilde{a}_\phi^\pm = a\left(Y_n + 2a\Delta \pm b\sqrt{2\Delta}\phi\right)
$$

where ϕ is either ζ or ρ. Here we use two correlated Gaussian random variables $\Delta\hat{W} \sim N(0;\Delta)$ and $\Delta\hat{Z} \sim N\left(0;\frac{1}{3}\Delta^3\right)$ with $E(\Delta\hat{W}\Delta\hat{Z}) = \frac{1}{2}\Delta^2$, together with two independent two-point distributed random variables ζ and ρ with

$$
P(\zeta = \pm 1) = P(\rho = \pm 1) = \frac{1}{2}.
$$

2 Richardson Extrapolation Methods

First in deterministic case ($b = 0$), Euler schemes is first order, so,

$$
y_N(\Delta) = x(T) + e(T)\Delta + O\left(\Delta^2\right)
$$

and

$$
y_{2N}\left(\frac{1}{2}\Delta\right) = x(T) + \frac{1}{2}e(T)\Delta + O\left(\Delta^2\right),
$$

in this way, we can expect,

$$
Z_N(\Delta) = 2y_{2N}\left(\frac{1}{2}\Delta\right) - y_N(\Delta)
$$

will be second order! this is called Richardson or Romberg extrapolation.

When approximating the expectation of a functional, say $E(f(X_T))$, the Euler scheme is also first order. We then define,

$$V^{\delta}_{g,2}(T) = 2E\left(g\left(Y^{\delta}(T)\right)\right) - E\left(g\left(Y^{2\delta}(T)\right)\right)$$

to achieve second order.

Further more, given order 2 weak approximation, we define,

$$V^{\delta}_{g,4}(T) = \frac{1}{21}\left[32E\left(g\left(Y^{\delta}(T)\right)\right) - 12E\left(g\left(Y^{2\delta}(T)\right)\right) + E\left(g\left(Y^{4\delta}(T)\right)\right)\right]$$

to achieve 4-th order. And given order 3 weak approximation, we define,

$$V^{\delta}_{g,6}(T) = \frac{1}{2905}\left[4032E\left(g\left(Y^{\delta}(T)\right)\right) - 1512E\left(g\left(Y^{2\delta}(T)\right)\right) + 448E\left(g\left(Y^{3\delta}(T)\right)\right) - 63E\left(g\left(Y^{4\delta}(T)\right)\right)\right]$$

to achieve 6-th order.

General Theory Evaluation δ can be generalized and changes of coefficients follows. In general, consider,

$$\delta_l = d_l \delta$$

for $l = 1, \dots, \beta + 1$ with

$$0 < d_1 < \cdots < d_{\beta+1} < \infty$$

an order 2β weak extrapolation is given by the expression (3.6)

$$V^{\delta}_{g,2\beta}(T) = \sum_{l=1}^{\beta+1} a_l E\left(g\left(Y^{\delta_1}(T)\right)\right)$$

where (if)

$$\sum_{l=1}^{\beta+1} a_l = 1$$

and

$$\sum_{l=1}^{\beta+1} a_l \left(d_l\right)^{\gamma} = 0$$

for each $\gamma = \beta, \dots, 2\beta - 1$.

3 Predictor-Corrector Method

3.1 Implicit Weak Method

To improve the stability of weak schemes, we also consider implicit version.

Implicit Euler The simplest implicit weak scheme is the implicit Euler scheme, which in the general multi-dimensional case $d, m = 1, 2, \ldots$ has the form

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\Delta + \sum_{j=1}^{m} b^j\left(\tau_n, Y_n\right)\Delta\hat{W}^j$$

where the $\Delta\hat{W}^j$ for $j = 1, \ldots, m$ and $n = 1, 2, \ldots$ are independent two-point distributed random variables with

$$P\left(\Delta\hat{W}^j = \pm\sqrt{\Delta}\right) = \frac{1}{2}$$

We can also form a family of implicit Euler schemes

$$\begin{aligned} Y_{n+1} = Y_n + & \left\{(1-\alpha)a\left(\tau_n, Y_n\right) + \alpha a\left(\tau_{n+1}, Y_{n+1}\right)\right\}\Delta \\ & + \sum_{j=1}^{m} b^j\left(\tau_n, Y_n\right)\Delta\hat{W}^j \end{aligned}$$

Note again, implicit Euler is A-stable and fully implicit Euler is even not weak consistent.

Implicit Order 2.0 scheme The implicit Taylor order 2.0 scheme and its RK version,

$$\begin{aligned} Y_{n+1} = Y_n + & \frac{1}{2}\left\{a\left(\tau_{n+1}, Y_{n+1}\right) + a\right\}\Delta \\ & + \sum_{j=1}^{m} b^j\Delta\hat{W}^j + \frac{1}{2}\sum_{j=1}^{m} L^0 b^j\Delta\hat{W}^j\Delta \\ & + \frac{1}{2}\sum_{j_1, j_2=1}^{m} L^{j_1} b^{j_2}\left(\Delta\hat{W}^{j_1}\Delta\hat{W}^{j_2} + V_{j_1, j_2}\right) \end{aligned}$$

and

$$\begin{aligned} & Y_{n+1} = Y_n + \tfrac{1}{2}\left(a + a\left(Y_{n+1}\right)\right)\Delta \\ & + \tfrac{1}{4}\textstyle\sum_{j=1}^{m}\Big[b^j\left(\bar{R}_+^j\right) + b^j\left(\bar{R}_-^j\right) + 2b^j \\ & \quad + \textstyle\sum_{r=1, r\neq j}^{m}\left(b^j\left(\bar{U}_+^r\right) + b^j\left(\bar{U}_-^r\right) - 2b^j\right)\Delta^{-1/2}\Big]\Delta\hat{W}^j \\ & + \tfrac{1}{4}\textstyle\sum_{j=1}^{m}\Bigg[\left(b^j\left(\bar{R}_+^j\right) - b^j\left(\bar{R}_-^j\right)\right)\left\{\left(\Delta\hat{W}^j\right)^2 - \Delta\right\} \\ & + \textstyle\sum_{r=1, r\neq j}^{m}\left(b^j\left(\bar{U}_+^r\right) - b^j\left(\bar{U}_-^r\right)\right)\left\{\Delta\hat{W}^j\Delta\hat{W}^r + V_{r,j}\right\}\Bigg]\Delta^{-1/2} \end{aligned}$$

with supporting values

$$\bar{R}^j_{\pm} = Y_n + a\Delta \pm b^j\sqrt{\Delta}$$

and

$$\bar{U}^j_{\pm} = Y_n \pm b^j\sqrt{\Delta},$$

are A-stable.

3.2 Constructing Predict-Corrector schemes

The idea is to use $\frac{a\left(\bar{Y}_{n+1}\right)+a)}{2}$ to replace $a(Y_{n+1})$ in the implicit scheme.

Oder 1 scheme We can construct a family of order 1.0 weak predictor-corrector methods with corrector,

$$Y_{n+1} = Y_n + \left\{\alpha a\left(\tau_{n+1}, \bar{Y}_{n+1}\right) + (1-\alpha)a\left(\tau_n, Y_n\right)\right\}\Delta + \sum_{j=1}^{m} b^j\left(\tau_n, Y_n\right)\Delta\hat{W}^j$$

for $\alpha, \in [0,1]$, with predictor

$$\bar{Y}_{n+1} = Y_n + a\Delta + \sum_{j=1}^{m} b^j\Delta\hat{W}^j$$

where the $\Delta\hat{W}^j$ are as we defined before.

Order 2 Scheme In the autonomous 1 -dimensional scalar noise case, $d = m = 1$, a possible order 2.0 weak predictor-corrector method has corrector (5.7)

$$Y_{n+1} = Y_n + \frac{1}{2}\left\{a\left(\bar{Y}_{n+1}\right) + a\right\}\Delta + \Psi_n$$

with

$$\Psi_n = b\Delta\hat{W} + \frac{1}{2}bb'\left\{(\Delta\hat{W})^2 - \Delta\right\} + \frac{1}{2}\left(ab' + \frac{1}{2}b^2b''\right)\Delta\hat{W}\Delta$$

and predictor

$$\begin{aligned}\bar{Y}_{n+1} = Y_n + &a\Delta + \Psi_n \\ &+ \frac{1}{2}a'b\Delta\hat{W}\Delta + \frac{1}{2}\left(aa' + \frac{1}{2}a''b^2\right)\Delta^2\end{aligned}$$

where the $\Delta\hat{W}$ are $N(0;\Delta)$ Gaussian or three-point distributed with

$$P(\Delta\hat{W} = \pm\sqrt{3\Delta}) = \frac{1}{6}, \quad P(\Delta\hat{W} = 0) = \frac{2}{3}.$$

4 Project VI (due June 4)

Consider the SDE:

$$dX_t = aX_t dt + bX_t dW_t,$$

a, b constants, has exact solution:

$$X_t = X_0 \exp\{(a - \frac{b^2}{2})t + bW_t\}.$$

Let $X_0 = 1$, $a = 1.5$, $b = 1$.

(1) Solve by implicit Euler, implicit Milstein:

$$\begin{aligned} Y_{n+1} &= Y_n + a(Y_{n+1})\Delta + b\Delta W \\ &+\frac{1}{2}bb'[(\Delta W)^2 - \Delta], \end{aligned} \tag{4.1}$$

and implicit Runge-Kutta:

$$\begin{aligned} Y_{n+1} &= Y_n + a(Y_{n+1})\Delta + b\Delta W \\ &+ \frac{1}{2\sqrt{\Delta}}(b(Y_n^*) - b)[(\Delta W)^2 - \Delta)], \end{aligned} \tag{4.2}$$

$Y_n^* = Y_n + a\Delta + b\sqrt{\Delta}$. Generate 10000 sample solutions at $\Delta = 2^{-m}$, $m = 1:8$, and plot $\log_2$ of the root mean square error at $t = 1$ vs. $\log_2 \Delta$.

(2) Solve by weak Euler and the simplified weak Taylor:

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta\hat{W} + \frac{1}{2}bb'((\Delta\hat{W})^2 - \Delta) \\ &+\frac{1}{2}(a'b + ab' + \frac{1}{2}b''b^2)\Delta\hat{W}\Delta \\ &+\frac{1}{2}(aa' + \frac{1}{2}a''b^2)\Delta^2. \end{aligned} \tag{4.3}$$

The $\Delta\hat{W}$ for both schemes is a normal random variable $N(0,\Delta)$ (mean zero, variance Δ). Generate 10000 sample solutions at $\Delta = 2^{-m}$, $m = 1:8$, and plot $\log_2$ of the absolute errors of the first and second moments of $X(1)$ and $Y(1)$ vs. $\log_2 \Delta$.

(3) Apply Richardson extrapolation to method in (2) with same Δ and plot $\log_2$ of the absolute errors of the first and second moments of $X(1)$ and $Y(1)$ vs. $\log_2 \Delta$.