

# Lecture 15: Weak Taylor Approximation

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## Abstract

Introducing weak schemes based on Ito-Taylor expansion and the convergence theorem.

## 1 Weak Euler Scheme

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n, \quad (1.1)$$

with initial data  $Y_0 = X_0$ .

*Weak Approximation* is for approximating the measure (or moments) related to the Ito SDE solution  $X(t)$ . One could replace  $\Delta W_n$  by a simple two-point distributed r.v  $\Delta \tilde{W}_n$  with:

$$Prob(\Delta \tilde{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}.$$

To study weak convergence of approximation, introduce space  $H^{(l)}$  for functions of  $x$ ,  $l \in (0, 1) \cup (1, 2) \cup (2, 3)$ .  $H^{(l)}$  consists of  $u(x)$  such that  $\partial_x^s u$  is Hölder continuous with exponent  $l - [l]$ ,  $[l]$  integral part of  $l$ ,  $s$  an integer  $\leq l$ . Hölder norm of a function  $v(x)$  is:

$$\|v\| = \sup_{x \neq x'} \frac{|v(x) - v(x')|}{|x - x'|^{l-[l]}}.$$

The  $H^{(l)}$  norm is:

$$\|u\|_l = \|\partial_x^{[l]} u\| + \sum_{s \leq l} \sup |u^{(s)}(x)|.$$

The convergence of Euler weak approximation is:

**Theorem 1.1** *Let  $X(t)$  be Ito SDE solution over  $[0, T]$ ,  $a(x)$ ,  $b(x) \in H^{(l)}$ , and let  $Y^\delta(t)$  be Euler approximation with time step  $\delta$ . For any function  $g \in H^{(l+2)}$ :*

$$|E(g(X(T))) - E(g(Y^\delta(T)))| \leq K\delta^{x^{(l)}},$$

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$$\chi(l) = \begin{cases} l/2, & \text{if } l \in (0, 1), \\ 1/(3-l), & \text{if } l \in (1, 2), \\ 1, & \text{if } l \in (2, 3), \end{cases} \quad (1.2)$$

and  $K$  independent of  $l$ .

**Remark 1.1** *If the coefficients  $a$  and  $b$  are slightly more differentiable than twice, the weak convergence is first order. When  $l = 1$ , namely, coefficients are Lipschitz, weak convergence is order 0.5.*

**Remark 1.2** *In proof of lecture 7, we can only verify  $c(\delta) \leq \delta^2$  which yields 0.5 order weak convergence.*

## 1.1 Convergence of Weak Euler

Let  $f = f(t, x)$  be a Hölder continuous function of exponent  $l$  in  $x \in R^1$ ,  $l/2$  in  $t \in [0, T]$ , such functions form the Hölder space  $H_T^{(l)}$ . Let  $Y^\delta(t)$  be the Euler approximate solution of Ito SDE solution  $X(t)$  starting from same initial data  $X_0 = Y_0$ . It is assumed to be interpolated exactly with fixing  $a$  and  $b$  at grid point, when  $t$  is a not grid point. The noise increment  $\Delta\tilde{W}$  satisfies:

$$E(|\Delta\tilde{W}|^3) + |E(\Delta\tilde{W})^2 - \Delta| \leq K\Delta^2. \quad (1.3)$$

**Lemma 1.1** *Suppose drift and diffusion  $a$  and  $b$  are bounded, then for any  $\eta \in (0, 1)$ , there is a positive constant  $K_\eta$  such that:*

$$|E(f(s, Y^\delta(s)) - f(\tau_{n_s}, Y_{n_s}^\delta) | A_{\tau_{n_s}})| \leq K_\eta \|f\|_T^{(l)} \delta^{\chi(l)}, \quad (1.4)$$

$s \in [0, T]$ ,  $l \in [\eta, 1) \cup (1, 2) \cup (2, 3)$ ,  $\chi$  is defined in (1.2).

Proof: let  $w_\epsilon(x) = \frac{1}{\epsilon} w(\frac{x}{\epsilon})$ , the mollifier ( $w > 0$ ,  $\int w dx = 1$ ), define:

$$f^{h,\epsilon} = h^{-1} \int_t^{t+h} \int f(\min(u, T), y) w_\epsilon(x - y) dy du,$$

then:

$$\sup_{t,x} |f(t, x) - f^{h,\epsilon}(t, x)| \leq \|f\|_T^{(l)} (h^{\min(l/2, 1)} + \epsilon^{\min(l, 1)}), \quad (1.5)$$

$$\sup_{t,x} |\partial_x^i f^{h,\epsilon}(t, x)| \leq K \|f\|_T^{(l)} \epsilon^{\min(l-i, 0)}, \quad (1.6)$$

$$\sup_{t,x} |\partial_t f^{h,\epsilon}(t, x)| \leq K \|f\|_T^{(l)} h^{\min(-1+l/2, 0)}, \quad (1.7)$$

$i = 1, 2$ , min with 1 in (1.5) is due to first differencing of left had side; integer derivatives in (1.6)-(1.7) reduce exponent by 1.

We replace  $f$  by  $f^{h,\epsilon}$  and estimate errors.

$$\begin{aligned} & |E(f(s, Y^\delta(s)) - f(\tau_{n_s}, Y_{n_s}^\delta) | A_{\tau_{n_s}})| \\ & \leq 2 \sup_{t,x} |f(t, x) - f^{h,\epsilon}(t, x)| \\ & + |E(f^{h,\epsilon}(s, Y^\delta(s)) - f^{h,\epsilon}(\tau_{n_s}, Y_{n_s}^\delta) | A_{\tau_{n_s}})| \end{aligned} \quad (1.8)$$

Noticing that  $Y^\delta(s)$  is exact interpolation, the second term of (1.8) is estimated by Ito formula thanks to (1.5)-(1.7), skipping superscript  $\delta$  on  $Y^\delta$ :

$$\begin{aligned} & \leq |E(\int_{\tau_{n_s}}^s [\partial_t f^{h,\epsilon}(u, Y(u)) + \frac{1}{2}b(\tau_{n_s}, Y_{n_s})f_{xx}^{h,\epsilon}(u, Y(u)) \\ & + a(\tau_{n_s}, Y_{n_s})f_x^{h,\epsilon}(u, Y(u))] du | A_{\tau_{n_s}})| \\ & \leq K \|f\|_T^{(l)} (h^{\min(-1+l/2,0)} + \epsilon^{\min(l-2,0)}) \delta. \end{aligned} \quad (1.9)$$

So:

$$\begin{aligned} & |E(f(s, Y^\delta(s)) - f(\tau_{n_s}, Y_{n_s}^\delta) | A_{\tau_{n_s}})| \\ & \leq K \|f\|_T^{(l)} [\inf_{h \in (0,1)} (h^{\min(l/2,1)} + h^{\min(-1+l/2,0)}) \delta) \\ & + \inf_{\epsilon \in (0,1)} (\epsilon^{\min(l,1)} + \epsilon^{\min(l-2,0)}) \delta], \\ & \leq K_\eta \|f\|_T^{(l)} \delta^\chi, \end{aligned} \quad (1.10)$$

proof is finished.

**Proof of Theorem 1.1** Let:

$$L_0 = \partial_t + a(x)\partial_x + \frac{1}{2}b(x)\partial_{xx},$$

there is unique solution of final value problem:

$$L_0 v = 0, \quad v(T, x) = g(x), \quad (1.11)$$

such that:

$$\|v\|_T^{(l+2)} \leq K \|g\|^{(l+2)}, \quad (1.12)$$

and by Ito:

$$E(v(0, X_0)) = E(v(T, X_T)) = E(g(X_T)).$$

It follows by Ito formula and triangle inequality:

$$\begin{aligned}
& |E(g(X_T)) - E(g(Y(T)))| \\
= & |E(v(0, X_0)) - E(v(T, Y(T)))| = |E(v(T, Y(T))) - E(v(0, Y_0))| \\
= & |E(\int_0^T [\frac{1}{2}b(Y_{n_s})v_{xx} + a(Y_{n_s})v_x + v_t - L_0v](s, Y(s))ds)| \\
\leq & \int_0^T |E([b(Y_{n_s}) - b(Y(s))]v_{xx}(s, Y(s)))|ds \\
& + \int_0^T |E([a(Y_{n_s}) - a(Y(s))]v_x(s, Y(s)))|ds \\
\leq & \int_0^T |E(b(Y_{n_s})v_{xx}(\tau_{n_s}, Y_{n_s}) - b(Y(s))v_{xx}(s, Y(s))|A_{\tau_{n_s}})| \\
& + |E(b(Y_{n_s})[v_{xx}(\tau_{n_s}, Y_{n_s}) - v_{xx}(s, Y(s))]|A_{\tau_{n_s}})| ds \\
& + \dots
\end{aligned}$$

.... refer to similar terms on drift. Note that  $bv_{xx}$ ,  $v_{xx}$ ,  $av_x$ ,  $v_x$  all belong to  $H_T^{(l)}$  due to (1.12). Applying the lemma, we prove the weak convergence theorem of the Euler method.

## 2 Higher Order Weak Schemes

### 2.1 Order 2 Weak Schemes

Adding all of the double stochastic integrals from Ito-Taylor expansions gives the order 2 weak scheme:

$$\begin{aligned}
Y_{n+1} = & Y_n + a\Delta + b\Delta W + \frac{1}{2}bb'((\Delta W)^2 - \Delta) \\
& + a'b\Delta Z + \frac{1}{2}(aa' + \frac{1}{2}a''b^2)\Delta^2 \\
& + (ab' + \frac{1}{2}b''b^2)(\Delta W\Delta - \Delta Z)
\end{aligned} \tag{2.13}$$

$\Delta Z = \int_0^\Delta W_s ds$ . Here  $\Delta W$  and  $\Delta Z$  are generated jointly by mapping independent unit Gaussians  $U_i$ ,  $i = 1, 2$ .

$$\Delta W = U_1\sqrt{\Delta}, \quad \Delta Z = \frac{1}{2}\Delta^{3/2}(U_1 + \frac{1}{\sqrt{3}}U_2).$$

Simplified weak schemes are constructed by replacing  $\Delta W$  by a similarly distributed

$\Delta\hat{W}$ , and  $\Delta Z$  by  $\frac{1}{2}\Delta\hat{W}\Delta$  to approximate  $E(\Delta Z\Delta W) = \Delta^2/2$ :

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta\hat{W} + \frac{1}{2}bb'((\Delta\hat{W})^2 - \Delta) \\ &\quad + \frac{1}{2}(a'b + ab' + \frac{1}{2}b''b^2)\Delta\hat{W}\Delta \\ &\quad + \frac{1}{2}(aa' + \frac{1}{2}a''b^2)\Delta^2, \end{aligned} \tag{2.14}$$

where  $\Delta\hat{W}$  satisfies the moment condition:

$$E(|\Delta\hat{W}|^5) + |E((\Delta\hat{W})^2) - \Delta| + |E((\Delta\hat{W})^4) - 3\Delta^2| \leq K\Delta^3, \tag{2.15}$$

One may choose  $\hat{W}$  as  $N(0, \Delta)$ , or 3-point random variable taking  $\pm\sqrt{3\Delta}$  with prob 1/6 each, and zero with prob 2/3.

**General Multi-dimensional case** In the general multi-dimensional case  $d, m = 1, 2, \dots$  the  $k$  th component of the order 2.0 weak Taylor scheme takes the form

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + a^k\Delta + \frac{1}{2}L^0a^k\Delta^2 \\ &\quad + \sum_{j=1}^m \{b^{k,j}\Delta W^j + L^0b^{k,j}I_{(0,j)} + L^j a^k I_{(j,0)}\} \\ &\quad + \sum_{j_1, j_2=1}^m L^{j_1} b^{k, j_2} I_{(j_1, j_2)} \end{aligned} \tag{2.16}$$

For weak convergence we can substitute simpler random variables the multiple Ito integrals. In this way we obtain from (2.16) the following simplified order 2.0 weak Taylor scheme with  $k$  th component

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + a^k\Delta + \frac{1}{2}L^0a^k\Delta^2 \\ &\quad + \sum_{j=1}^m \left\{ b^{k,j} + \frac{1}{2}\Delta(L^0b^{k,j} + L^j a^k) \right\} \Delta\hat{W}^j \\ &\quad + \frac{1}{2} \sum_{j_1, j_2=1}^m L^{j_1} b^{k, j_2} \left( \Delta\hat{W}^{j_1} \Delta\hat{W}^{j_2} + V_{j_1, j_2} \right) \end{aligned}$$

Here the  $\Delta\hat{W}^j$  for  $j = 1, 2, \dots, m$  are independent random variables satisfying (2.15) and the  $V_{j_1, j_2}$  are independent two-point distributed random variables with

$$P(V_{j_1, j_2} = \pm\Delta) = \frac{1}{2}$$

for  $j_2 = 1, \dots, j_1 - 1$ ,

$$V_{j_1, j_1} = -\Delta$$

and

$$V_{j_1, j_2} = -V_{j_2, j_1}$$

## 2.2 Order 3 Schemes

Consider  $d = m = 1$ ,

$$\begin{aligned} Y_{n+1} = & Y_n + a\Delta + b\Delta W + L^0 a I_{(0,0)} + L^1 a I_{(1,0)} + L^0 b I_{(0,1)} + L^1 b I_{(1,1)} \\ & + L^0 L^0 a I_{(0,0,0)} + L^0 L^1 a I_{(0,1,0)} + L^1 L^0 a I_{(1,0,0)} + L^1 L^1 a I_{(1,1,0)} \\ & + L^0 L^0 b I_{(0,0,1)} + L^0 L^1 b I_{(0,1,1)} + L^1 L^0 b I_{(1,0,1)} + L^1 L^1 b I_{(1,1,1)} \end{aligned}$$

By comparing moments, we propose,

$$\begin{aligned} Y_{n+1} = & Y_n + a\Delta + b\Delta\tilde{W} + \frac{1}{2}L^1 b \left\{ (\Delta\tilde{W})^2 - \Delta \right\} \\ & + L^1 a \Delta\tilde{Z} + \frac{1}{2}L^0 a \Delta^2 + L^0 b \left\{ \Delta\tilde{W}\Delta - \Delta\tilde{Z} \right\} \\ & + \frac{1}{6} (L^0 L^0 b + L^0 L^1 a + L^1 L^0 a) \Delta\tilde{W}\Delta^2 \\ & + \frac{1}{6} (L^1 L^1 a + L^1 L^0 b + L^0 L^1 b) \left\{ (\Delta\tilde{W})^2 - \Delta \right\} \Delta \\ & + \frac{1}{6} L^0 L^0 a \Delta^3 + \frac{1}{6} L^1 L^1 b \left\{ (\Delta\tilde{W})^2 - 3\Delta \right\} \Delta\tilde{W} \end{aligned}$$

where  $\Delta\tilde{W}$  and  $\Delta\tilde{Z}$  are correlated Gaussian random variables with

$$\Delta\tilde{W} \sim N(0; \Delta), \quad \Delta\tilde{Z} \sim N\left(0; \frac{1}{3}\Delta^3\right)$$

and covariance

$$E(\Delta\tilde{W}\Delta\tilde{Z}) = \frac{1}{2}\Delta^2.$$

## 3 General Rule and Convergence

In general, a weak order  $\beta = 1, 2, 3, \dots$  scheme needs all of the multiple Ito integrals from the Ito-Taylor expansion in the set  $\Gamma_\beta = \{\alpha : l(\alpha) \leq \beta\}$ . Here  $l$  is the length of the index  $\alpha$ . Note that is different from the strong scheme index set  $A_\gamma$  which also depends on the number of zeros in the index  $n(\alpha)$ .

**Theorem 3.1** *Let  $Y^\delta$  be a time discrete approximation of an autonomous Ito process  $X$  corresponding to a time discretization  $(\tau)_\delta$ , such that all moments of the initial value  $X_0$  exist, that is*

$$E(|X_0|^i) < \infty$$

*for  $i = 1, 2, \dots$ , and such that  $Y_0^\delta$  converges weakly with order  $\beta$  to  $X_0$  as  $\delta \rightarrow 0$  for some fixed  $\beta = 1.0, 2.0, \dots$ . Assume that  $a(x), b(x)$  are  $C^{2(\beta+1)}$  and all derivatives up to  $2(\beta+1)$  have polynomial growth in large  $x$ . In addition, suppose that for each  $p = 1, 2, \dots$  there exist constants  $K < \infty$  and  $r \in \{1, 2, \dots\}$ , which do not depend on  $\delta$ , such that for each  $q \in \{1, \dots, p\}$*

$$E\left(\max_{0 \leq n \leq n_T} |Y_n^\delta|^{2q} \mid \mathcal{A}_0\right) \leq K \left(1 + |Y_0^\delta|^{2r}\right)$$

*and  $E(|Y_{n+1}^\delta - Y_n^\delta|^{2q} \mid \mathcal{A}_{\tau_n}) \leq K \left(1 + \max_{0 \leq k \leq n} |Y_k^\delta|^{2r}\right) (\tau_{n+1} - \tau_n)^q$  for  $n = 0, 1, \dots, n_T - 1$ , and such that*

$$\begin{aligned} & \left| E\left(\prod_{h=1}^l (Y_{n+1}^{\delta, p_h} - Y_n^{\delta, p_h}) - \prod_{h=1}^l \left(\sum_{\alpha \in \Gamma_\beta \setminus \{v\}} f_\alpha^{p_h}(\tau_n, Y_n^\delta) I_{\alpha, \tau_n, \tau_{n+1}}\right) \mid \mathcal{A}_{\tau_n}\right) \right| \\ & \leq K \left(1 + \max_{0 \leq k \leq n_T} |Y_k^\delta|^{2r}\right) \delta^\beta (\tau_{n+1} - \tau_n) \end{aligned} \quad (3.17)$$

*for all  $n = 0, 1, \dots, n_T - 1$  and  $(p_1, \dots, p_l) \in \{1, \dots, d\}^l$ , where  $l = 1, \dots, 2\beta + 1$  and  $Y^{\delta, p_h}$  denotes the  $p_h$  th component of  $Y^\delta$ . Then the time discrete approximation  $Y^\delta$  converges weakly with order  $\beta$  as  $\delta \rightarrow 0$  to the Ito process  $X$  at time  $T$ .*

A straight forward corollary follows,

**Corollary 3.1** *Let  $X(t)$  be an autonomous Ito SDE solution over  $[0, T]$ . Let  $Y^\delta$  be solution of a weak scheme of order  $\beta = 1, 2, 3, \dots$ , with exact Brownian increment. Then for any function  $g \in C^{2(\beta+1)}$  whose derivatives up to  $2(\beta+1)$  have polynomial growth in large  $x$ ,*

$$|E(g(X(T))) - E(g(Y^\delta(T)))| \leq K_g \delta^\beta,$$

$K_g$  independent of  $\delta$ .

Note, left hand side of (3.17) is zero with exact Brownian increment.