

# Lecture 12: Prior Estimate

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## Abstract

Prior estimate for multiple stochastic integrals.

Recall **Doob Inequality**: A martingale  $X = \{X_t, t \geq 0\}$  with finite  $p$  th- moment ( $p > 1$ ) satisfies,

$$E \left( \sup_{0 \leq s \leq t} |X_s|^p \right) \leq \left( \frac{p}{p-1} \right)^p E(|X_t|^p).$$

Cauchy-Schwartz inequality,

$$\left| \int f(x)g(x) dx \right|^2 \leq \int |f(x)|^2 dx \int |g(x)|^2 dx.$$

## 1 Moments of Multiple Stochastic Integrals

**First Moments Lemma:** Let  $\alpha \in \mathcal{M} \setminus \{v\}$  with  $l(\alpha) \neq n(\alpha)$ , let  $f \in \mathcal{H}_\alpha$  and let  $\rho$  and  $\tau$  be two stopping times with  $t_0 \leq \rho \leq \tau \leq T < \infty$ , w.p.1. Then

$$E(I_\alpha[f(\cdot)]_{\rho, \tau} | \mathcal{A}_\rho) = 0, \quad w.p.1 \quad (1.1)$$

**A Mean-Square Lemma:** Let  $\rho \leq \tau \leq \rho + \delta \leq T$ , then:

$$\begin{aligned} E \left( \sup_{s \in [\rho, \tau]} |I_\alpha[g]_{\rho, s}|^2 | A_\rho \right) &\leq 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_\rho^\tau R_{\rho, s} ds \\ R_{\rho, s} &= E(\sup_{\rho \leq t \leq s} |g(t)|^2 | A_\rho) < \infty. \end{aligned} \quad (1.2)$$

*Proof:* Induction on  $\alpha$ . First  $\alpha = (0)$ ,  $l(\alpha) = 1$ ,  $n(\alpha) = 1$ .

$$\begin{aligned} E \left( \sup_{s \in [\rho, \tau]} \left| \int_\rho^s g(z) dz \right|^2 | A_\rho \right) &\leq E \left( \delta \int_\rho^s |g(z)|^2 dz | A_\rho \right) \\ &\leq 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_\rho^\tau R_{\rho, s} ds. \end{aligned} \quad (1.3)$$

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$\alpha = (1)$ ,  $l(\alpha) = 1$ ,  $n(\alpha) = 0$ :

$$\begin{aligned}
& E \left( \sup_{s \in [\rho, \tau]} \left| \int_{\rho}^s g(z) dW_z \right|^2 \middle| A_{\rho} \right) \\
& \leq 4E \left( \left| \int_{\rho}^{\tau} g(z) dW_z \right|^2 \middle| A_{\rho} \right) \\
& \leq 4 \int_{\rho}^{\tau} E(|g(z)|^2 | A_{\rho}) dz \\
& \leq 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_{\rho}^{\tau} R_{\rho,s} ds.
\end{aligned} \tag{1.4}$$

The factor 4 comes from Doob's inequality for martingales.

$\alpha = (\alpha_1, \dots, \alpha_{k+1})$ ,  $l(\alpha) = k + 1$ .

Case I:  $\alpha_{k+1} = 0$ .

$$\begin{aligned}
& E \left( \sup_{s \in [\rho, \tau]} \left| \int_{\rho}^s I_{\alpha-}[g]_{\rho,z} dz \right|^2 \middle| A_{\rho} \right) \\
& \leq E \left( \sup_{s \in [\rho, \tau]} (s - \rho) \int_{\rho}^s |I_{\alpha-}[g]_{\rho,z}|^2 dz \middle| A_{\rho} \right) \\
& \leq E \left( \delta \int_{\rho}^{\tau} |I_{\alpha-}[g]_{\rho,z}|^2 dz \middle| A_{\rho} \right) \\
& \leq \delta^2 E \left( \sup_{s \in [\rho, \tau]} |I_{\alpha-}[g]_{\rho,s}|^2 \middle| A_{\rho} \right),
\end{aligned} \tag{1.5}$$

by induction:

$$\begin{aligned}
& \leq \delta^2 \delta^{l(\alpha-)+n(\alpha-)-1} 4^{l(\alpha-)-n(\alpha-)} \int_{\rho}^{\tau} R_{\rho,z} dz \\
& \leq 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_{\rho}^{\tau} R_{\rho,z} dz.
\end{aligned} \tag{1.6}$$

Case II:  $\alpha_{k+1} \neq 0$ . By Doob, induction:

$$\begin{aligned}
& E \left( \sup_{s \in [\rho, \tau]} \left| \int_{\rho}^s I_{\alpha-}[g]_{\rho, z} dW_z \right|^2 \middle| \mathcal{A}_{\rho} \right) \\
& \leq 4 \sup_{s \in [\rho, \tau]} E \left( \left| \int_{\rho}^s I_{\alpha-}[g]_{\rho, z} dW_z \right|^2 \middle| \mathcal{A}_{\rho} \right) \\
& \leq 4 \sup_{s \in [\rho, \tau]} \int_{\rho}^s E \left( |I_{\alpha-}[g]_{\rho, z}|^2 \middle| \mathcal{A}_{\rho} \right) dz \\
& \leq 4\delta E \left( \sup_{s \in [\rho, \tau]} |I_{\alpha-}[g]_{\rho, s}|^2 \middle| \mathcal{A}_{\rho} \right) \\
& \leq 4\delta 4^{l(\alpha-) - n(\alpha-)} \delta^{l(\alpha-) + n(\alpha-) - 1} \int_{\rho}^{\tau} R_{\rho, z} dz \\
& \leq 4^{l(\alpha) - n(\alpha)} \delta^{l(\alpha) + n(\alpha) - 1} \int_{\rho}^{\tau} R_{\rho, z} dz.
\end{aligned} \tag{1.7}$$

**Estimates of Higher Moments (Rough):** With the same setting,

$$\left( E \left( |I_{\alpha}[g(\cdot)]_{\rho, \tau}|^{2q} \middle| \mathcal{A}_{\rho} \right) \right)^{1/q} \leq (2(2q-1)e^T)^{l(\alpha) - n(\alpha)} (\tau - \rho)^{l(\alpha) + n(\alpha)} R \tag{1.8}$$

where

$$R = \left( E \left( \sup_{\rho \leq s \leq \tau} |g(s)|^{2q} \middle| \mathcal{A}_{\rho} \right) \right)^{1/q} \tag{1.9}$$

## 2 Estimate of a Multiple Ito Integral

### 2.1 The estimate

Let  $\alpha = (\alpha_1, \alpha_2, \dots) \neq v$ ,  $v$  the empty index,  $\delta$  the time step of discretization over  $[0, T]$ ,  $\tau_n$ 's the uniform discrete time steps,  $g$  a right continuous adapted process. Let:

$$R_{0, u} = E \left( \sup_{s \in [0, u]} |g(s)|^2 \middle| \mathcal{A}_0 \right) < \infty, \tag{2.10}$$

$$F_t^{\alpha} = E \left( \sup_{z \in [0, t]} \left| \sum_{n=0}^{n_z-1} I_{\alpha}[g(\cdot)]_{\tau_n, \tau_{n+1}} + I_{\alpha}[g(\cdot)]_{\tau_{n_z}, z} \right|^2 \middle| \mathcal{A}_0 \right). \tag{2.11}$$

Then,

$$F_t^{\alpha} \leq t \delta^{2(l(\alpha)-1)} \int_0^t R_{0, u} du, \quad \text{if } l(\alpha) = n(\alpha), \tag{2.12}$$

and

$$F_t^{\alpha} \leq 4^{l(\alpha) - n(\alpha) + 2} \delta^{l(\alpha) + n(\alpha) - 1} \int_0^t R_{0, u} du, \quad l(\alpha) \neq n(\alpha), \tag{2.13}$$

where

$$n_z := \max\{n \in N \mid \tau_n \leq z\}. \quad (2.14)$$

## 2.2 Remark

The case  $l = n$  is the deterministic Riemann integrals, we see that the total error is  $O(\delta^{l-1})$  while local error of each term is  $O(\delta^l)$ . When  $l \neq n$ , total error is  $O(\delta^{\frac{l+n-1}{2}})$ . Each term conditioned locally is  $O(\delta^{(l+n)/2}) = O(\delta^{\frac{n'}{2}+n})$ .

Let  $A_\gamma$  be the indices for discretizations of order  $\gamma$  in the truncated Ito-Taylor expansion.

$$\begin{aligned} A_{1/2} &= \{v, (0), (1)\}, \\ A_1 &= \{v, (0), (1), (1, 1)\}, \\ A_{1.5} &= \{v, (0), (1), (1, 1), (0, 1), (1, 0), (0, 0), (1, 1, 1)\}, \\ A_2 &= A_{1.5} \cup \{(0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1, 1)\}. \end{aligned} \quad (2.15)$$

A general expression for  $A_\gamma$  is:

$$A_\gamma = \{l(\alpha) + n(\alpha) \leq 2\gamma, \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}. \quad (2.16)$$

## 2.3 Proof

When  $l(\alpha) = n(\alpha)$ ,

$$\begin{aligned} F_t^\alpha &= E\left(\sup_{z \in [0, t]} \left| \int_0^z I_{\alpha-}[g(\cdot)]_{\tau_{n_u}, u} du \right|^2 \middle| A_0\right) \\ &\leq t \cdot E\left(\sup_{z \in [0, t]} \int_0^z |I_{\alpha-}[g(\cdot)]_{\tau_{n_u}, u}|^2 du \middle| A_0\right) \\ &\leq t \int_0^t E\left(E\left(\sup_{s \in [\tau_{n_u}, u]} |I_{\alpha-}[g(\cdot)]_{\tau_{n_u}, s}|^2 \middle| A_{\tau_{n_u}}\right) \middle| A_0\right) du. \end{aligned} \quad (2.17)$$

By Lemma in (1.2):

$$\begin{aligned} F_t^\alpha &\leq t 4^{l(\alpha-)-n(\alpha-)} \delta^{l(\alpha-)+n(\alpha-)-1} \int_0^t E\left(\int_{\tau_{n_u}}^u R_{\tau_{n_u}, s} ds \middle| A_0\right) du \\ &\leq t \delta^{l(\alpha-)+n(\alpha-)} \int_0^t E(R_{\tau_{n_u}, u} \middle| A_0) du \\ &\leq t \delta^{2(l(\alpha)-1)} \int_0^t R_{0, u} du. \end{aligned} \quad (2.18)$$

When  $l(\alpha) \neq n(\alpha)$ :

**Case I:**  $n(\alpha-) = n(\alpha) - 1$ .

$$F_t^\alpha \leq 2E\left(\sup_{z \in [0,t]} \left| \sum_{n=0}^{n_z-1} I_\alpha[g]_{\tau_n, \tau_{n+1}} \right|^2 \middle| A_0\right) + 2E\left(\sup_{z \in [0,t]} |I_\alpha[g]_{\tau_{n_z}, z}|^2 \middle| A_0\right). \quad (2.19)$$

For the first term, use Doob inequality and Lemma in (1.2):

$$\begin{aligned} & E\left(\sup_{z \in [0,t]} \left| \sum_{n=0}^{n_z-1} I_\alpha[g]_{\tau_n, \tau_{n+1}} \right|^2 \middle| A_0\right) \\ & \leq \sup_{z \in [0,t]} 4E\left[\left| \sum_{n=0}^{n_z-1} I_\alpha[g]_{\tau_n, \tau_{n+1}} \right|^2 \middle| A_0\right) \\ & \leq \sup_{z \in [0,t]} 4E\left(\left| \sum_{n=0}^{n_z-2} I_\alpha[g]_{\tau_n, \tau_{n+1}} \right|^2 \right. \\ & \quad \left. + 2 \sum_{n=0}^{n_z-2} I_\alpha[g]_{\tau_n, \tau_{n+1}} \cdot E[I_\alpha[g]_{\tau_{n_z-1}, \tau_{n_z}} \middle| A_{\tau_{n_z-1}}] \right. \\ & \quad \left. + E[|I_\alpha[g]_{\tau_{n_z-1}, \tau_{n_z}}|^2 \middle| A_{\tau_{n_z-1}}] \middle| A_0\right) \end{aligned} \quad (2.20)$$

$$\begin{aligned} & \leq \sup_{z \in [0,t]} 4E\left(\left| \sum_{n=0}^{n_z-2} I_\alpha[g]_{\tau_n, \tau_{n+1}} \right|^2 + \right. \\ & \quad \left. 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_{\tau_{n_z-1}}^{\tau_{n_z}} R_{\tau_{n_z-1}, u} du \middle| A_0\right). \end{aligned} \quad (2.21)$$

Iterating (2.21):

$$\begin{aligned} & \leq \sup_{z \in [0,t]} 4E\left(\left| \sum_{n=0}^{n_z-3} I_\alpha[g]_{\tau_n, \tau_{n+1}} \right|^2 + \right. \\ & \quad \left. 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_{\tau_{n_z-2}}^{\tau_{n_z-1}} R_{\tau_{n_z-2}, u} du \right. \\ & \quad \left. + 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_{\tau_{n_z-1}}^z R_{\tau_{n_z-1}, u} du \middle| A_0\right) \\ & \leq \sup_{z \in [0,t]} 4E\left(4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_0^z R_{0, u} du \middle| A_0\right) \\ & \leq 4^{l(\alpha)-n(\alpha)+1} \delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0, u} du. \end{aligned} \quad (2.22)$$

The 2nd term of (2.19) is bounded as:

$$\begin{aligned}
& E\left[\sup_{z \in [0,t]} |I_\alpha[g]_{\tau_{nz},z}|^2 \middle| A_0\right] \\
&= E\left(\sup_{z \in [0,t]} \left| \int_{\tau_{nz}}^z I_{\alpha-}[g]_{\tau_{nz},u} du \right|^2 \middle| A_0\right) \\
&\leq E\left(\sup_{z \in [0,t]} (z - \tau_{nz}) \int_{\tau_{nz}}^z |I_{\alpha-}[g]_{\tau_{nz},u}|^2 du \middle| A_0\right) \\
&\leq \delta \int_0^t E\left(E\left(\sup_{s \in [\tau_{nu},u]} |I_{\alpha-}[g]_{\tau_{nu},s}|^2 \middle| A_{\tau_{nu}}\right) \middle| A_0\right) du \\
&\leq \delta 4^{l(\alpha-)-n(\alpha-)} \int_0^t E\left(\int_{\tau_{nu}}^u R_{\tau_{nu},s} ds \delta^{l(\alpha-)+n(\alpha-)-1} \middle| A_0\right) du \\
&\leq 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0,u} du.
\end{aligned} \tag{2.23}$$

Note in the estimate we only need  $4^{l(\alpha)-n(\alpha)+2}$ .

**Case II:**  $l(\alpha) \neq n(\alpha)$ ,  $n(\alpha-) = n(\alpha)$ .

$$\begin{aligned}
F_t^\alpha &= E\left(\sup_{z \in [0,t]} \left| \int_0^z I_{\alpha-}[g]_{\tau_{nu},u} dW_u \right|^2 \middle| A_0\right) \\
&\leq 4 \sup_{z \in [0,t]} E\left[\left| \int_0^z I_{\alpha-}[g]_{\tau_{nu},u} dW_u \right|^2 \middle| A_0\right] \\
&\leq 4 \sup_{z \in [0,t]} \int_0^z E\left(E|I_{\alpha-}[g]_{\tau_{nu},u}|^2 \middle| A_{\tau_{nu}}\right) \middle| A_0\right) du \\
&\leq 4 \int_0^t E\left(E\left(\sup_{s \in [\tau_{nu},u]} |I_{\alpha-}[g]_{\tau_{nu},s}|^2 \middle| A_{\tau_{nu}}\right) \middle| A_0\right) du \\
&\leq 4 4^{l(\alpha-)-n(\alpha-)} \int_0^t E\left(\int_{\tau_{nu}}^u R_{\tau_{nu},s} ds \delta^{l(\alpha-)+n(\alpha-)-1} \middle| A_0\right) du \\
&\leq 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0,u} du.
\end{aligned} \tag{2.24}$$

Proof is complete.