Lecture 10: Poisson random variables

Statistics 251

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Midterm Announcement

- Time: Oct.26 10:20AM-11:10AM+20 min to upload
- Test paper: Available on Gradescope at 10:19 AM
- Venue: Zoom Conference Room for lectures
- Remark: Join with your name on UID. Open your camera. Sign your name ahead. No need of calculator!
- Important notice: Do not discuss with anybody else, except with me, until Oct. 27 10:20 AM CDT.

Where are we?

Motivation

Consider the following random variables:

- The number of plane crashes in a year.
- The number of calls to a call center in an hour.
- The number of goals during a 90 minute soccer match.
- The number of raindrops which hit an umbrella in a minute.

Motivation

Consider the following random variables:

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In all of these examples, we can divide the time interval into small **increments** and add up the count during each interval. The count in one interval is **approximately independent** of other intervals.

Reminder on e

Recall that

$$e = \lim_{n\to\infty} (1 + 1/n)^n$$

and

$$e^\lambda = \lim_{n\to\infty} (1 + \lambda/n)^n$$

and

$$e^{-\lambda} = \lim_{n\to\infty} (1 - \lambda/n)^n.$$

Poisson random variables

Let $\lambda > 0$ be a number, and let n be huge (e.g. 10^6).

Suppose I take n tosses of a coin that comes up heads with probability $p = \frac{\lambda}{n}$. How many heads do I expect?

- Answer: $n \cdot p = \lambda$.

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What is the probability I get k heads?

- Binomial(n, p), so

$$\begin{aligned}
\mathbb{P}(X = k) &= \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{\lambda^k}{n^k} (1 - \lambda/n)^{n-k} \\
&\approx \frac{\lambda^k}{k!} e^{-\lambda} \quad (n \to \infty).
\end{aligned}$$

Poisson random variables

A **Poisson** random variable with parameter λ has

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Where are we?

Poisson normalization

A **Poisson** random variable with parameter λ has

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

How can we show $\sum_{k=0}^{\infty} \mathbb{P}(X = k) = 1$?

We have that

$$\sum_{k=0}^{\infty} \mathbb{P}(X = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1,$$

where we note by Taylor expansion that

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.$$

Poisson expectation

A **Poisson** random variable with parameter λ has

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

What is $\mathbb{E}[X]$? ($\approx$ Binomial$(n, \lambda/n)$, so expect λ)

We may compute that

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\mathbb{P}(X = k) = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda$$

Poisson variance

A **Poisson** random variable with parameter λ has

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

What is $\mathrm{Var}[X]$?

We may compute that

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \mathbb{P}(X = k) = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=0}^{\infty} k \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=0}^{\infty} (1 + (k-1)) \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} + \lambda^2 \sum_{k=0}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} e^{-\lambda} = \lambda^2 + \lambda.
\end{aligned}$$

This means $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda$.

Where are we?

Problems

A country has an average of 2 plane crashes per year. Is it reasonable to assume the number of crashes is Poisson with parameter 2?

Under this assumption, what is the probability of exactly 2 crashes? What about 0?

Problems

A city has an average of 5 major earthquakes a century. If the number of earthquakes is Poisson, what is the probability there is at least an earthquake in a given decade?

An online poker site deals 1 million poker hands per day. Approximate the probability that there are exactly 2 royal flush hands in a given day.

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A city has an average of 5 major earthquakes a century. If the number of earthquakes is Poisson, what is the probability there is at least an earthquake in a given decade?

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Probability of royal flush: $\frac{\binom{4}{1}}{\binom{52}{5}} = 0.000154\%$.