Week 2

Statistics 251

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Where are we?

Equal likelihood

Inclusion-Exclusion

Conditional probability

Bayes' rule

Independence

Events of equal likelihood

Suppose a sample space S has N events. If each event is equally likely, what is the probability of each event?

- If each event has probability p, then $p \cdot N = 1$, so $p = \frac{1}{N}$

What is $\mathbb{P}(A)$ for a general subset $A \subset S$?

- A consists of $|A|$ disjoint events, so the total probability is $\frac{|A|}{|S|}$.

Events of equal likelihood

What is the probability that the sum of two dice rolls is 3?

What is the probability that exactly 4 of 8 coin tosses are heads?

Events of equal likelihood (problem session)

Roll 5 dice. What is the probability that exactly 2 of the dice show the number 1?

In a class of 60 students, what is the probability that none of them have student ID ending in 9?

Birthday paradox (problem session)

In a room of N people, what is the probability that some two of them have the same birthday?

What is the smallest N for which this probability is above $\frac{1}{2}$?

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Principle of inclusion-exclusion

Suppose we roll 2 dice and get two numbers a and b. What is the probability that either $a = b$ or a is even?

- Define $A = \{a = b\}$ and $B = \{a \text{ is even}\}$.
- $\mathbb{P}(A) = \frac{1}{6}$, $\mathbb{P}(B) = \frac{1}{2}$, and $\mathbb{P}(A \cap B) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Principle of inclusion-exclusion

More generally, suppose we have events $E_1, \ldots, E_n$. Then, we have

$$\mathbb{P}\Big(E_1 \cup \cdots \cup E_n\Big) = \sum_{i=1}^{n} \mathbb{P}(E_i) - \sum_{i<j} \mathbb{P}(E_i \cap E_j) + \sum_{i<j<k} \mathbb{P}(E_i \cap E_j \cap E_k) - \cdots .$$

Derangements

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- Let E_i be the probability that card i is in position i after the shuffle.
- Because all orderings are equally likely, $\mathbb{P}(E_i) = \frac{1}{n}$.
- For any $i_1, \ldots, i_k$, $\mathbb{P}(E_{i_1} \cap \cdots \cap E_{i_k}) = \frac{(n-k)!}{n!}$.

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- For any $i_1, \ldots, i_k$, $\mathbb{P}(E_{i_1} \cap \cdots \cap E_{i_k}) = \frac{(n-k)!}{n!}$.
- In the principle of inclusion-exclusion, we have $\binom{n}{k}$ such terms, which means

$$\mathbb{P}\Big(E_1 \cup \cdots \cup E_n\Big) = \binom{n}{1} \cdot \frac{(n-1)!}{n!} - \binom{n}{2} \cdot \frac{(n-2)!}{n!} + \binom{n}{3} \cdot \frac{(n-3)!}{n!} - \cdots = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

- The answer is $1 - \mathbb{P}(E_1 \cup \cdots \cup E_n) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots = \frac{1}{e}$.

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Definition of conditional probability

Let S be a sample space and $F \subset S$ a subset.

- Suppose a random event in S is drawn.
- The **conditional probability** of another event E *given* F is the probability that E happened given that F happened. Quantitatively, it is

$$\mathbb{P}(E \mid F) := \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

This is known as "the probability of E conditioned on F".

Multiplication rule

For events $E_1, \ldots, E_n$, we have

$$\mathbb{P}\Big(E_1 \cap \cdots \cap E_n\Big) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2 \mid E_1) \cdots \mathbb{P}(E_n \mid E_1 \cap E_2 \cap \cdots \cap E_{n-1}).$$

To have $E_1 \cap \cdots \cap E_n$, must first have E_1, then E_2 given E_1, then E_3 given $E_1 \cap E_2$, then E_4 given $E_1 \cap E_2 \cap E_3$, ...

Definition of conditional probability

The conditional probability $\mathbb{P}(- \mid F)$ satisfies axioms of a probability space:

- $0 \leq \mathbb{P}(E \mid F) \leq 1$
- $\mathbb{P}(S \mid F) = 1$
- $\mathbb{P}\Big(\bigcup_{i \in I} E_i \mid F\Big) = \sum_i \mathbb{P}(E_i \mid F)$ for E_i disjoint and I countable.

Intepretation of $\mathbb{P}(- \mid F)$:

- Probabilities of events outside F are set to 0;
- Probabilities of events inside F are multiplied by $\frac{1}{\mathbb{P}(F)}$.

Conditional probability

Graph the probability that a bus will arrive eventually, as a function of minutes to the scheduled arrival.

Conditional probability

Graph the probability that American Airlines will go bankrupt, given a sequence of events.

- Flights between US and China canceled
- Europe travel ban
- Federal airline bailout
- ...

Conditional probability

Let E_i be the probability that a roll of a die lies outside $\{1, \ldots, i\}$
What is $\mathbb{P}(E_4 \mid E_1 \cap E_2 \cap E_3)$?

Monty Hall problem

A TV show in the 1970s had the following game:

- There is a prize behind 1 of 3 doors, all equally likely.
- You point to a door, say A. The host opens one of the other two doors, say B, and shows you it does not have a prize.
- You get to open a door and claim what is behind it. Should you stay with A or switch to C?

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Recollection on conditional probability

Remember that the conditional probability of E given F is

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

Equivalently, we have

$$\mathbb{P}(E \cap F) = \mathbb{P}(F) \cdot \mathbb{P}(E \mid F).$$

Law of total probability

Suppose we want to compute $\mathbb{P}(E)$. For another event F, we have

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c) \\ &= \mathbb{P}(E \mid F) \cdot \mathbb{P}(F) + \mathbb{P}(E \mid F^c) \cdot \mathbb{P}(F^c).\end{aligned}$$

Law of total probability: Example

Example: $D =$ "have disease", $+ =$ "test positive". Suppose $\mathbb{P}(D) = p$, $\mathbb{P}(+ \mid D) = 0.9$, and $\mathbb{P}(+ \mid D^c) = 0.1$. We have

$$\begin{aligned}\mathbb{P}(+) = \mathbb{P}(+ \mid D) \cdot \mathbb{P}(D) + \mathbb{P}(+ \mid D^c) \cdot \mathbb{P}(D^c) \\ = 0.9p + 0.1(1-p) = 0.1 + 0.8p.\end{aligned}$$

What we really care about is $\mathbb{P}(D \mid +)$, which is

$$\frac{\mathbb{P}(D \cap +)}{\mathbb{P}(+)} = \frac{\mathbb{P}(+ \mid D) \cdot \mathbb{P}(D)}{\mathbb{P}(+)} = \frac{0.9p}{0.9p + 0.1(1-p)}.$$

Law of total probability

If events $F_1, F_2, \ldots$ partition the sample space, then

$$\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \mid F_i) \cdot \mathbb{P}(F_i).$$

Proof: If F_i are disjoint and parition S, then $E \cap F_i$ are disjoint and partition E. So we have

$$\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}\Big(E \cap S\Big) \\
&= \mathbb{P}\Big(E \cap (F_1 \cup F_2 \cup \cdots)\Big) \\
&= \mathbb{P}\Big((E \cap F_1) \cup (E \cup F_2) \cup \cdots\Big) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(E \cap F_i) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(E \mid F_i) \cdot \mathbb{P}(F_i).
\end{aligned}$$

Bayes' rule

Bayes' rule: By $\mathbb{P}(B \mid A) \cdot \mathbb{P}(A) = \mathbb{P}(A \cap B) = \mathbb{P}(A \mid B) \cdot \mathbb{P}(B)$, we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)}\mathbb{P}(A).$$

Bayes' rule

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$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)} \mathbb{P}(A).$$

Interpretation: Start with estimate $\mathbb{P}(A)$ for A. After receiving new information, perform a **Bayesian update** to restrict the sample space to B.

- $\frac{\mathbb{P}(B|A)}{\mathbb{P}(B)}$ measures how strong the evidence is
- If $\mathbb{P}(B \mid A) = 0$, A and B are mutually exclusive.
- We have $\frac{\mathbb{P}(B|A)}{\mathbb{P}(B)} \leq \frac{1}{\mathbb{P}(A)}$, with equality if and only if $\mathbb{P}(A \cap B) = \mathbb{P}(B)$.

Bayesian updating

Draw a card at random from a deck. Define the events

$$A = \{\text{card is ace of spades}\}$$
$$B = \{\text{suit of card is from spades}\}.$$

To start, we know that $\mathbb{P}(A) = \frac{1}{52}$. If we now know that B occurred, we may update by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)}\mathbb{P}(A) = \frac{1}{1/4} \cdot \frac{1}{52} = \frac{1}{13}.$$

We may update as further information emerges about the card.

Bayesian updating in the real world...

We can assign probabilites to events which have not yet happened:

$$\mathbb{P}(\text{Biden wins election again})$$
$$\mathbb{P}(\text{Chicago Cubs win the World Series})$$
$$\mathbb{P}(\text{stock prices will go up this year}).$$

According to Thomas Bayes:

$$\mathbb{P}(A) := \{\text{value of right to get \$1 if event occurs}\}.$$

This creates philosophical questions:

- Does this "value" have a well-defined price?
- How is $\mathbb{P}(A)$ defined when there are no enforceable financial contracts?
- Can we use this interpretation (and Bayes' rule) in everyday reasoning?

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Independent events

Events E and F are **independent** if $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$.

- Equivalent formulation: $\mathbb{P}(E \mid F) = \mathbb{P}(E)$
- Equivalent formulation: $\mathbb{P}(F \mid E) = \mathbb{P}(F)$

Intuitively: knowing E occurred does not change the likelihood that F occurred, and vice versa.

Independent events

Are the following events independent?

- Flip two coins: $\{$first coin H$\}$ and $\{$second coin H$\}$
- Flip two coins: $\{$first coin H$\}$ and $\{$total of 2 Hs$\}$
- Flip two coins: $\{$first coin H$\}$ and $\{$odd number of H$\}$
- Draw two cards from same deck: $\{$first card K$\}$ and $\{$second card Q$\}$
- Randomly find a patient from a hospital: $\{$age 0-15$\}$ and $\{$hospitalized for infection$\}$
- Randomly find a patient from a hospital: $\{$first name starts with A$\}$ and $\{$hospitalized for infection$\}$

Independence of multiple events

Events $E_1, \ldots, E_n$ are **independent** if for each $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ we have

$$\mathbb{P}\Big(E_{i_1} \cap \cdots \cap E_{i_k}\Big) = \mathbb{P}(E_{i_1})\mathbb{P}(E_{i_2}) \cdots \mathbb{P}(E_{i_k}).$$

Implies statements like $\mathbb{P}(E_1 \cap E_2 \mid E_3 \cap E_4 \cap E_5) = \mathbb{P}(E_1 \cap E_2)$.

Pairwise independence does not imply independence:

- Consider {first coin H}, {second coin H}, {odd number of H}

Independence: examples

Shuffle 4 cards labeled $1, 2, 3, 4$. Let

$$E_{i,j} = \{\text{card } i \text{ comes before card } j\}.$$

Is $E_{1,2}$ independent of $E_{3,4}$?

Is $E_{1,2}$ independent of $E_{1,3}$?

Independence: examples

What is $\mathbb{P}(E_{1,2} \mid E_{1,3})$?

What is $\mathbb{P}(E_{1,7} \mid E_{1,2} \cap E_{1,3} \cap \cdots \cap E_{1,6})$?

Independence: examples

Suppose you roll a fair dice repeatedly. What is the probability that you get a 6 for the first time, on the 3rd roll?

Independence: examples (problem session)

Suppose you draw three card at random from a standard deck.
If we get a number (`2`, `3`, `...`,`10`), that value is the number of points earned.
If we get a `J`, `Q`, or `K`, then we earn 10 points. An `A` earns 0 points.

What is the prob. that you earn 10 points, then 5 points, then 10 points?