Lecture 8: Ito-Taylor Expansion I

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Abstract

Derivation of Ito-Taylor Expansion, notation of stochastic integrals, coefficient functions

1 Deterministic Analogue

Consider ODE:

$$\frac{d}{dt}X = a(X), \quad X(t) = X_0 + \int_0^t a(X(s))\,ds, \tag{1.1}$$

chain rule on any f:

$$\frac{d}{dt}f(X) = a(X)f'(X) \equiv Lf(X), \tag{1.2}$$

or:

$$f(X) = f(X_0) + \int_0^t Lf(X)(s)\,ds. \tag{1.3}$$

Applying (1.3) to $a(X(s))$ in (1.1), we get:

$$\begin{aligned} X(t) &= X_0 + \int_0^t \left(a(X_0) + \int_0^s La(X)(z)\,dz \right) ds \\ &= X_0 + a(X_0)\int_0^t ds + \int_0^t \int_0^s La(X)(z)\,dz\,ds. \end{aligned} \tag{1.4}$$

Repeating once more:

$$X(t) = X_0 + a(X_0)\int_0^t ds + La(X_0)\int_0^t \int_0^s dz\,ds + R, \tag{1.5}$$

with:

$$R = \int_0^t \int_0^{s_3} \int_0^{s_2} L^2 a(X)(s_1) ds_1\,ds_2\,ds_3.$$

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Continuing this process n times, one recovers Taylor expansion in integral form:

$$\begin{aligned} f(X)(t) &= f(X_0) + \sum_{j=1}^{n} \frac{t^j}{j!}(L^j f)(X_0) \\ &\quad + \int_0^t \int_0^{s_{n+1}} \cdots \int_0^{s_2} (L^{n+1} f)(X)(s_1)\, ds_1 \cdots ds_{n+1}. \end{aligned} \tag{1.6}$$

2 Ito-Taylor Expansion

Consider Ito eqn:

$$X_t = X_0 + \int_0^t a(X_s)\, ds + \int_0^t b(X_s)\, dW_s, \tag{2.7}$$

Ito formula gives:

$$f(X_t) = f(X_0) + \int_0^t (L^0 f)(X_s)\, ds + \int_0^t (L^1 f)(X_s)\, dW_s, \tag{2.8}$$

$$L^0 = a\frac{d}{dx} + \frac{1}{2}b^2\frac{d^2}{dx^2},$$

$$L^1 = b\frac{d}{dx}.$$

Substitute (2.8) into (2.7):

$$\begin{aligned} X_t &= X_0 \\ &\quad + \int_0^t \left(a(X_0) + \int_0^s (L^0 a)(X_z)\, dz + \int_0^s (L^1 a)(X_z)\, dW_z \right) ds \\ &\quad + \int_0^t \left(b(X_0) + \int_0^s (L^0 b)(X_z) dz + \int_0^s (L^1 b)(X_z) dW_z \right) dW_s \\ &= X_0 + a(X_0)\int_0^t ds + b(X_0)\int_0^t dW_s + R, \end{aligned} \tag{2.9}$$

remainder:

$$\begin{aligned} R &= \int_0^t \int_0^s (L^0 a)(X_z) dz ds + \int_0^t \int_0^s (L^1 a)(X_z) dW_z ds \\ &+ \int_0^t \int_0^s (L^0 b)(X_z) dz dW_s + \int_0^t \int_0^s (L^1 b)(X_z)\, dW_z dW_s. \end{aligned} \tag{2.10}$$

Continuing once more by applying Ito formula to $L^1 b(X_z)$:

$$\begin{aligned} X_t &= X_0 + a(X_0)\int_0^t ds + b(X_0)\int_0^t dW_s \\ &\quad + (L^1 b)(X_0)\int_0^t \int_0^s dW_z dW_s + R_1, \end{aligned} \tag{2.11}$$

remainder:

$$
\begin{aligned}
R_1 \quad = \quad & \int_0^t \int_0^s (L^0 a)(X_z) dz ds + \int_0^t \int_0^s (L^1 a)(X_z) dW_z ds \\
& + \int_0^t \int_0^s (L^0 b)(X_z) dz dW_s + \int_0^t \int_0^s \int_0^z (L^0 L^1 b)(X_u) du dW_z dW_s \\
& + \int_0^t \int_0^s \int_0^z (L^1 L^1 b)(X_u) dW_u dW_z dW_s. \qquad (2.12)
\end{aligned}
$$

This is called Ito-Taylor formula, in priciple, one could continue given enough smoothness of a and b, to generate an expansion. The remainder involves multiple stochastic Ito integrals.

3 Shortened Notations

3.1 Multi-indices

We shall call a row vector

$$\alpha = (j_1, j_2, \ldots, j_l)$$

where

$$j_i \in \{0, 1, \ldots, m\}$$

for $i \in \{1, 2, \ldots, l\}$ and $m = 1, 2, 3, \ldots$, a multi-index of length

$$l := l(\alpha) \in \{1, 2, \ldots\}$$

Here m will denote the number of components of the Wiener process under consideration. For completeness we denote by v the multi-index of length zero, that is with

$$l(v) := 0$$

Thus, for example,

$$l((1,0)) = 2 \text{ and } l((1,0,1)) = 3.$$

In addition, we shall write $n(\alpha)$ for the number of components of a multi-index which are equal to 0 . For example,

$$n((1,0,1)) = 1, \quad n((0,1,0)) = 2, \quad n((0,0)) = 2.$$

We denote the set of all multi-indices by $\mathcal{M}$, so

$$\mathcal{M} = \left\{ (j_1, j_2, \ldots, j_l) : j_i \in \{0, 1, \ldots, m\}, i \in \{1, \ldots, l\}, \text{ for } l = 1, 2, 3, \ldots \right\} \cup \{v\}. \quad (3.13)$$

Given $\alpha \in \mathcal{M}$ with $l(\alpha) \geq 1$, we write $-\alpha$ and $\alpha-$ for the multi-index in $\mathcal{M}$ obtained by deleting the first and the last component, respectively, of α. Thus

$$-(1,0) = (0),(1,0)- = (1)$$
$$-(0,1,1) = (1,1),(0,1,1)- = (0,1).$$

Finally, for any two multi-indices $\alpha = (j_1, j_2, \dots, j_k)$ and $\bar{\alpha} = (\bar{j}_1, \bar{j}_2, \dots, \bar{j}_l)$ we introduce an operation $*$ on $\mathcal{M}$ by

$$\alpha * \bar{\alpha} = (j_1, j_2, \dots, j_k, \bar{j}_1, \bar{j}_2, \dots, \bar{j}_l)$$

the multi-index formed by adjoining the two given multi-indices. We shall call this the concatenation operation. For example, for $\alpha = (0,1,2)$ and $\bar{\alpha} = (1,3)$ we have

$$\alpha * \bar{\alpha} = (0,1,2,1,3) \text{ and } \bar{\alpha} * \alpha = (1,3,0,1,2)$$

3.2 Multiple Ito Integrals

Given integrable condition, we can recursively define,

$$I_\alpha[f(\cdot)]_{\rho,\tau} := \begin{cases} f(\tau) & : \ l = 0 \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s} ds & : \ l \geq 1 \text{ and } j_l = 0 \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s} dW_s^{j_1} & : \ l \geq 1 \text{ and } j_l \geq 1, \end{cases}$$

where $l(\alpha) = l$, for example,

$$\begin{aligned} I_0[f]_{0,t} &= \int_0^t f(s)ds; \\ I_1[f]_{0,t} &= \int_0^t f(s)dW_s; \\ I_{(0,1)}[f]_{0,t} &= \int_0^t \int_0^{s_2} f(s_1)ds_1\, dW_{s_2} \end{aligned} \tag{3.14}$$

and $I_j = I_j[1]$. Then Ito-Taylor expansion up to two layer integrals is:

$$\begin{aligned} X_t &= X_0 + aI_0 + bI_1 + (aa' + \frac{1}{2}b^2 a'')I_{(0,0)} \\ &\quad +[ab' + \frac{1}{2}b^2 b'']I_{(0,1)} + ba' I_{(1,0)} + bb' I_{(1,1)} + \cdots, \end{aligned} \tag{3.15}$$

dots mean higher layered integrals.

Some Calculations

- For convenience, we write

$$I_{\alpha,t} = I_\alpha[1]_{0,t}, \quad W_t^0 = t.$$

- Let $j_1, \ldots, j_l \in \{0, 1, \ldots, m\}$ and $\alpha = (j_1, \ldots, j_l) \in \mathcal{M}$ where $l = 1, 2, 3, \ldots$ Then

$$W_t^j I_{\alpha,t} = \sum_{i=0}^{l} I_{(j_1,\ldots,j_i,j,j_{i+1},\ldots,j_l),t} + \sum_{i=1}^{l} \mathbf{1}_{\{j_i=j\neq 0\}} I_{(j_1,\ldots,j_{i-1},0,j_{i+1},\ldots,j_l),t} \tag{3.16}$$

for all $t \geq 0$.

Sketch of Proof: By Ito formula of function like $f(X,Y) = XY$,

$$\begin{aligned} W_t^j I_{\alpha,t} = I_{(j),t} I_{\alpha,t} &= \int_0^t I_{\alpha,s} dI_{(j),s} + \int_0^t I_{(j),s} I_{\alpha-,s} dW_s^{j_l} + \mathbf{1}_{\{j_l=j\neq 0\}} \int_0^t I_{\alpha-,s} ds & (3.17)\\ &= I_{(j_1,\ldots,j_l,j),t} + \int_0^l W_s^j I_{\alpha-,s} dW_s^{j_l} + \mathbf{1}_{\{j_l=j\neq 0\}} I_{(j_1,\ldots,j_{l-1},0),t} & (3.18) \end{aligned}$$

Now, consider $W_s^j I_{\alpha-,s}$ by induction.

- (Corollary) Suppose that $\alpha = (j_1, \ldots, j_l)$ with $j_1 = \cdots = j_l = j \in \{0,, \ldots, m\}$ where $l \geq 2$. Then for $t \geq 0$

$$I_{\alpha,t} = \begin{cases} \frac{1}{l!} t^l & : \quad j = 0 \\ \frac{1}{l} \left(W_t^j I_{\alpha-,t} - t I_{(\alpha-)-,t} \right) & : \quad j \geq 1 \end{cases}$$

Sketch of Proof: The case $j = 0$ follows from the usual deterministic integration rule. For $j \in \{1, \ldots, m\}$ the relation (3.16) gives

$$\begin{aligned} t I_{(\alpha-)-,t} &= \sum_{i=0}^{l-2} I_{(j_1,\ldots,j_i,0,j_{i+1},\ldots,j_{l-2}),t} \text{ and} \\ W_t^j I_{\alpha-,t} &= l I_{\alpha,t} + \sum_{i=1}^{l-1} I_{(j_1,\ldots,j_{i-1},0,j_{i+1},\ldots,j_{l-1}),t} \end{aligned}$$

Examples:

$$I_{(j,j),t} = \frac{1}{2!} \left(I_{(j),t}^2 - t \right) \tag{3.19}$$

$$I_{(j,j,j),t} = \frac{1}{3!} \left(I_{(j),t}^3 - 3t I_{(j),t} \right) \tag{3.20}$$

$$I_{(j,j,j,j),t} = \frac{1}{4!} \left(I_{(j),t}^4 - 6t I_{(j),t}^2 + 3t^2 \right) \tag{3.21}$$

4 Coefficient Functions

We shall write the diffusion operator for the Ito equation in d dimension defined with m dimension Wiener process as

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^l}$$

and for $j \in \{1, \ldots, m\}$ introduce the operator

$$L^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k}$$

For each $\alpha = (j_1, \ldots, j_l)$ and function f (which has enough regularity), we define recursively the Ito coefficient function

$$f_\alpha = \begin{cases} f & : \quad l = 0 \\ L^{j_1} f_{-\alpha} & : \quad l \geq 1 \end{cases}$$

Rmk: it is in the opposite order with the multiple Ito integral. But both of them are natural and intuitive to understand.

If the function f is not explicitly stated we shall always take it to be the identity function $f(t, x) \equiv x$. For example, in the 1-dimensional case $d = m = 1$ for $f(t, x) \equiv x$ we have

$$f_{(0)} = a, \quad f_{(1)} = b, \quad f_{(1,1)} = bb'$$

and

$$f_{(0,1)} = ab' + \frac{1}{2} b^2 b''$$

Here the prime $'$ denotes the ordinary or partial derivative with respect to the x variable, depending on whether or not the function being differentiated depends only on x or on both t and x.

Now note

$$\begin{gathered} f_{(0)} = a, \quad f_{(j_1)} = b^{j_1} \\ f_{(0,0)} = aa' + \sigma a'', \quad f_{(0,j_1)} = ab^{j_1\prime} + \sigma b^{j_1\prime\prime} \\ f_{(j_1,0)} = b^{j_1} a', \quad f_{(j_1,j_2)} = b^{j_1} b^{j_2\prime} \end{gathered} \tag{4.22}$$

given

$$\sigma = \frac{1}{2} \sum_{j=1}^{m} \left(b^j\right)^2 \tag{4.23}$$

Combining we have,

$$
\begin{aligned}
X_t &= X_0 + aI_0 + bI_1 + (aa' + \frac{1}{2}b^2a'')I_{(0,0)} \\
&\quad +[ab' + \frac{1}{2}b^2b'']I_{(0,1)} + ba'I_{(1,0)} + bb'I_{(1,1)} + \cdots \qquad (4.24)\\
&= X_0 + f_{(0)}I_0 + f_{(1)}I_1 + f_{(0,0)}I_{(0,0)} \\
&\quad +f_{(0,1)}I_{(0,1)} + f_{(1,1)}I_{(1,0)} + f_{(1,1)}I_{(1,1)} + \cdots\ , \qquad (4.25)
\end{aligned}
$$