Lecture 11: Strong Schemes with higher order

Zhongjian Wang*

Abstract

Constructing higher order strong schemes based on Ito-Taylor expansion and approximation of multiple stochastic integral.

Consider Ito Taylor expansion in the case $d = m = 1$ and the hierarchical set

$$\mathcal{A} = \{\alpha \in \mathcal{M} : l(\alpha) \leq 3\},$$

$$\begin{aligned}
X_t = X_{t_0} &+ aI_{(0)} + bI_{(1)} + \left(aa' + \frac{1}{2}b^2a''\right) I_{(0,0)} \\
&+ \left(ab' + \frac{1}{2}b^2b''\right) I_{(0,1)} + ba'I_{(1,0)} + bb'I_{(1,1)} \\
&+ \left[a\left(aa'' + (a')^2 + bb'a'' + \frac{1}{2}b^2a'''\right) + \frac{1}{2}b^2\left(aa''' + 3a'a''\right.\right. \\
&+ \left.\left.\left((b')^2 + bb''\right)a'' + 2bb'a'''\right) + \frac{1}{4}b^4a^{(4)}\right] I_{(0,0,0)} \\
&+ \left[a\left(a'b' + ab'' + bb'b'' + \frac{1}{2}b^2b'''\right) + \frac{1}{2}b^2\left(a''b' + 2a'b''\right.\right. \\
&\left.\left.+ab''' + \left((b')^2 + bb''\right)b'' + 2bb'b''' + \frac{1}{2}b^2b^{(4)}\right)\right] I_{(0,0,1)} \\
&+ \left[a\left(b'a' + ba''\right) + \frac{1}{2}b^2\left(b''a' + 2b'a'' + ba'''\right)\right] I_{(0,1,0)} \\
&+ \left[a\left((b')^2 + bb''\right) + \frac{1}{2}b^2\left(b''b' + 2bb'' + bb'''\right)\right] I_{(0,1,1)} \\
&+ b\left(aa'' + (a')^2 + bb'a'' + \frac{1}{2}b^2a'''\right) I_{(1,0,0)} \\
&+ b\left(ab'' + a'b' + bb'b'' + \frac{1}{2}b^2b'''\right) I_{(1,0,1)} \\
&+ b\left(a'b' + a''b\right) I_{(1,1,0)} + b\left((b')^2 + bb''\right) I_{(1,1,1)} + R. \qquad (0.1)
\end{aligned}$$

If one keeps the first three terms, one gets the Euler method accurate of order $O(t^{1/2})$.

*Department of Statistics, University of Chicago

1 Order 1 Milstein Scheme

By taking a noisy term $bb'I_{(1,1)}$, the accuracy goes up to order one. The Ito-Taylor expansion up to two layer integrals and order $O(t^{3/2})$ terms for autonomous diffusion process X_t:

$$\begin{aligned} X_t &= X_0 + aI_0 + bI_1 + (aa' + \frac{1}{2}b^2 a'')I_{(0,0)} \\ &+ [ab' + \frac{1}{2}b^2 b'']I_{(0,1)} + ba' I_{(1,0)} \\ &+ bb' I_{(1,1)} + b((b')^2 + bb'')I_{(1,1,1)} + \cdots , \end{aligned} \tag{1.2}$$

dots mean other higher layered integrals.

The Euler method is accurate of order $O(t^{1/2})$, and it is a truncation of (0.1) taking the first 3 terms. The Milstein method is constructed from (0.1) by taking an additional noisy term $bb'I_{(1,1)}$.

To evaluate the double integral $I_{(1,1)}$, recall the Ito to Stratonovich conversion formula:

$$\int_0^t h(W_s)dW_s = \int_0^t h(W_t) \circ dW_s - \frac{1}{2}\int_0^t h'(W_s)ds, \tag{1.3}$$

for any C^1 function h. Then:

$$\begin{aligned} I_{(1,1)} &= \int_0^t W_s dW_s \\ &= \int_0^t W_s \circ dW_s - \frac{1}{2}\int_0^t ds \\ &= \frac{1}{2}(W_t^2 - t). \end{aligned} \tag{1.4}$$

Milstein scheme:

$$Y_{n+1} = Y_n + a\Delta + b\Delta W + \frac{1}{2}bb'((\Delta W)^2 - \Delta), \tag{1.5}$$

Δ time step, ΔW Brownian increment from t_{n-1} to t_n. We shall show that the Milstein scheme is strongly convergent of order $\gamma = 1$ for $a \in C^1$, $b \in C^2$. It is *the stochastic extension of deterministic Euler preserving the order of accuracy.*

In the general multi-dimensional case with $d, m = 1, 2, \ldots$ the k th component of the Milstein scheme has the form,

$$Y_{n+1}^k = Y_n^k + a^k\Delta + \sum_{j=1}^m b^{k,j}\Delta W^j + \sum_{j_1,j_2=1}^m L^{j_1} b^{k,j_2} I_{(j_1,j_2)}. \tag{1.6}$$

In which, we know,

$$I_{(j_1,j_1)} = \frac{1}{2}\left\{\left(\Delta W^{j_1}\right)^2 - \Delta\right\}, \tag{1.7}$$

and for $j_1 \neq j_2$

$$I_{(j_1,j_2)} = J_{(j_1,j_2)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_1} dW_{s_2}^{j_1} dW_{s_1}^{j_2}$$
$$= \frac{1}{2} W_\Delta^{j_1} W_\Delta^{j_2} - \frac{1}{2}\left(a_{j_2,0} W_\Delta^{j_1} - a_{j_1,0} W_\Delta^{j_2}\right) + \pi \sum_{r=1}^{\infty} r \left(a_{j_1,r} b_{j_2,r} - b_{j_1,r} a_{j_2,r}\right). \tag{1.8}$$

It can be approximated by,

$$J_{(j_1,j_2)}^p = \Delta\left(\frac{1}{2}\xi_{j_1}\xi_{j_2} + \sqrt{\rho_p}\left(\mu_{j_1,p}\xi_{j_2} - \mu_{j_2,p}\xi_{j_1}\right)\right) \tag{1.9}$$

$$+ \frac{\Delta}{2\pi}\sum_{r=1}^{p}\frac{1}{r}\left(\zeta_{j_1,r}\left(\sqrt{2}\xi_{j_2} + \eta_{j_2,r}\right) - \zeta_{j_2,r}\left(\sqrt{2}\xi_{j_1} + \eta_{j_1,r}\right)\right) \tag{1.10}$$

where

$$\rho_p = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2} \tag{1.11}$$

and $\xi_j, \mu_{j,p}, \eta_{j,r}$ and $\zeta_{j,r}$ are independent $N(0;1)$ Gaussian random variables with

$$\xi_j = \frac{1}{\sqrt{\Delta}}\Delta W^j.$$

How to choose p?

In general, we shall examine the mean-square error between J_α^p and J_α. The most sensitive approximation is $J_{(j_1,j_2)}^p$ because the others are either identical to J_α or *their* *mean-square error can be estimated by a constant times* Δ^γ *for some* $\gamma \geq 3$. We have

$$
\begin{aligned}
&E\left(\left|J^p_{(j_1,j_2)} - J_{(j_1,j_2)}\right|^2\right) \\
&= \Delta^2 E\left(A^p_{j_1,j_2} - A_{j_1,j_2}\right)^2 \\
&= \Delta^2 E\left(\frac{\pi}{\Delta}\sum_{r=p+1}^{\infty} r\left(a_{j_1,r}b_{j_2,r} - b_{j_1,r}a_{j_2,r}\right)\right)^2 \\
&\text{(Given } j_1 \neq j_2\text{)} \\
&= \pi^2\left(\sum_{r=p+1}^{\infty} r^2 E\left(a_{j_1,r}b_{j_2,r} - b_{j_1,r}a_{j_2,r}\right)^2\right) \\
&\text{(As } E\left(a^2_{j,r}\right) = E\left(b^2_{j,r}\right) = \frac{\Delta}{2r^2\pi^2}\text{)} \\
&= \frac{\Delta^2}{2\pi^2}\sum_{r=p+1}^{\infty}\frac{1}{r^2} \\
&\leq \frac{\Delta^2}{2\pi^2}\int_p^{\infty}\frac{1}{u^2}du = \frac{\Delta^2}{2\pi^2 p}
\end{aligned}
$$

We know the truncation error in Ito-Taylor expansion is $\mathcal{O}(\Delta^{3/2})$, so we expect

$$
E\left(\left|J^p_{(j_1,j_2)} - J_{(j_1,j_2)}\right|^2\right) = \mathcal{O}(\Delta^3) \tag{1.12}
$$

which yields,

$$
p = p(\Delta) \geq \frac{K}{\Delta}. \tag{1.13}
$$

2 Order 1.5 strong scheme

For the method to be order 1.5, include noisy terms in Ito-Taylor expansion up to order $O(t^{3/2})$, and deterministic terms of order $O(t^2)$, to be precise,

$$
\begin{aligned}
X_t &= X_0 + aI_0 + bI_1 + (aa' + \frac{1}{2}b^2a'')I_{(0,0)} \\
&\quad + [ab' + \frac{1}{2}b^2b'']I_{(0,1)} + ba'I_{(1,0)} \\
&\quad + bb'I_{(1,1)} + b((b')^2 + bb'')I_{(1,1,1)} + \cdots,
\end{aligned} \tag{2.14}
$$

dots mean other higher layered integrals.

$$
\begin{aligned}
I_{(1,1,1)} &= \int_0^t dW_s \int_0^s dW_{s_2} \int_0^{s_2} dW_{s_1} \\
&= \int_0^t dW_s \, \frac{1}{2}(W_s^2 - s) \\
&= \frac{1}{6}W_t^3 - \frac{1}{2}\int_0^t W_s ds - \frac{1}{2}\int_0^t s dW_s, \qquad (2.15)
\end{aligned}
$$

by the Ito to Stratonovich conversion formula:

$$
\int_0^t h(W_s)dW_s = \int_0^t h(W_t) \circ dW_s - \frac{1}{2}\int_0^t h'(W_s)ds, \qquad (2.16)
$$

for any C^1 function h. The last two terms add up to $-\frac{1}{2}tW_t$, hence:

$$
I_{(1,1,1)} = \frac{1}{3!}(W_t^3 - 3tW_t). \qquad (2.17)
$$

Order 1.5 scheme is:

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\Delta + b\Delta + \frac{1}{2}bb'((\Delta W)^2 - \Delta) \\
&\quad + a'b\Delta Z + \frac{1}{2}(aa' + b^2 a''/2)\Delta^2 \\
&\quad + (ab' + b^2 b''/2)(\Delta W \Delta - \Delta Z) \\
&\quad + \frac{1}{3!}b(bb'' + (b')^2)((\Delta W)^2 - 3\Delta)\Delta W, \qquad (2.18)
\end{aligned}
$$

$$
\Delta Z = I_{(1,0)} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dW_{s_1} ds_2, \qquad (2.19)
$$

with properties:
(1) $E(\Delta Z) = 0$,
(2) variance $E((\Delta Z)^2) = \Delta^3/3$, covariance $E(\Delta Z \Delta W) = \frac{\Delta^2}{2}$.

In fact ($t = \Delta$):

$$
\begin{aligned}
Var(\Delta Z) &= E((\int_0^t W_s \, ds)^2) = E(\int_0^t \int_0^t W_s W_s' ds ds') \\
&= \int_0^t \int_0^t \min(s, s') ds ds' \\
&= \int_0^t ds \, (\int_0^s + \int_s^t) \min(s, s') ds' \\
&= \int_0^t ds \, (s^2/2 + (t-s)s) \\
&= t^3/3 = \Delta^3/3.
\end{aligned}
$$

$$
\begin{aligned}
E(\Delta Z \Delta W) &= \lim E(\sum_{i=1}^{N} W(j\delta)\delta W(\Delta)) \\
&= \lim E(\sum_{i=1}^{N} (j\delta)\delta \\
&= \lim_{\delta\to 0, \delta N=\Delta} \delta^2 N(N+1)/2 = \Delta^2/2.
\end{aligned}
$$

The pair $(\Delta W, \Delta Z)$ can then be generated by a pair of unit Gaussian r.v (U_1, U_2) as:

$$
\begin{aligned}
\Delta W &= U_1\sqrt{\Delta}, \\
\Delta Z &= \frac{1}{2}\Delta^{3/2}(U_1 + U_2/\sqrt{3}),
\end{aligned}
\tag{2.20}
$$

3 Order 2 Scheme

Including order $O(t^2)$ terms in Ito-Taylor expansion, one could derive 2nd order accurate schemes. For simplicity, it is better to write it through Stratonovich-Taylor expansion derived in the same way as Ito-Taylor except the drift coefficient a is modified to $\underline{a} = a - \frac{1}{2}bb'$. The second order scheme is:

$$
\begin{aligned}
Y_{n+1} &= Y_n + \underline{a}\Delta + b\Delta W \\
&+ \frac{1}{2}bb'(\Delta W)^2 + b\underline{a}'\Delta Z \\
&+ \frac{1}{2}\underline{a}\underline{a}'\Delta^2 + \underline{a}b'(\Delta W\Delta - \Delta Z) \\
&+ \frac{1}{3!}b(bb')'(\Delta W)^3 + \frac{1}{4!}b(b(bb')')'(\Delta W)^4 \\
&+ \underline{a}(bb')'J_{(0,1,1)} + b(\underline{a}b')'J_{(1,0,1)} \\
&+ b(b\underline{a}')'J_{(1,1,0)},
\end{aligned}
\tag{3.21}
$$

the $J_{(0,1,1)}$ etc are defined same as $I_{(0,1,1)}$ only with the integration in the sense of Stratonovich. The ΔW and ΔZ are the same as in 1.5 scheme.

$$
\Delta W = J^p_{(1)} = \sqrt{\Delta}\zeta_1, \quad \Delta Z = J^p_{(1,0)} = \frac{1}{2}\Delta\left(\sqrt{\Delta}\zeta_1 + a_{1,0}\right)
$$

$$
J^p_{(1,0,1)} = \frac{1}{3!}\Delta^2\zeta_1^2 - \frac{1}{4}\Delta a_{1,0}^2 + \frac{1}{\pi}\Delta^{3/2}\zeta_1 b_1 - \Delta^2 B^p_{1,1}
$$

$$
J^p_{(0,1,1)} = \frac{1}{3!}\Delta^2\zeta_1^2 - \frac{1}{2\pi}\Delta^{3/2}\zeta_1 b_1 + \Delta^2 B^p_{1,1} - \frac{1}{4}\Delta^{3/2}a_{1,0}\zeta_1 + \Delta^2 C^p_{1,1}
$$

$$
J^p_{(1,1,0)} = \frac{1}{3!}\Delta^2\zeta_1^2 + \frac{1}{4}\Delta a_{1,0}^2 - \frac{1}{2\pi}\Delta^{3/2}\zeta_1 b_1 + \frac{1}{4}\Delta^{3/2}a_{1,0}\zeta_1 - \Delta^2 C^p_{1,1}
$$

with

$$a_{1,0} = -\frac{1}{\pi}\sqrt{2\Delta}\sum_{r=1}^{p}\frac{1}{r}\xi_{1,r} - 2\sqrt{\Delta\rho_p}\mu_{1,p}, \quad \rho_p = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}$$

$$b_1 = \sqrt{\frac{\Delta}{2}}\sum_{r=1}^{p}\frac{1}{r^2}\eta_{1,r} + \sqrt{\Delta\alpha_p}\phi_{1,p}, \quad \alpha_p = \frac{\pi^2}{180} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^4}$$

$$B_{1,1}^{p} = \frac{1}{4\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}\left(\xi_{1,r}^2 + \eta_{1,r}^2\right)$$

$$C_{1,1}^{p} = -\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\neq l}}^{p}\frac{r}{r^2-l^2}\left(\frac{1}{l}\xi_{1,r}\xi_{1,l} - \frac{l}{r}\eta_{1,r}\eta_{1,l}\right)$$

where $\zeta_1, \xi_{1,r}\eta_{1,r}, \mu_{1,p}$ and $\phi_{1,p}$ for $r = 1, \ldots, p$ and $p = 1, 2, \ldots$ denote independent standard Gaussian random variables.