Lecture 7: Expectation

Statistics 251

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Lecture Outline

Definition of expectation

Expectation of functions of a random variable

Future applications

Where are we?

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Expectation of a discrete random variable

Recall: A random variable X is a function $X : S \to \mathbb{R}$. If it is discrete, it has probability mass function $p_X(a) = \mathbb{P}(X = a)$.

The **expectation** of X is

$$\mathbb{E}[X] := \sum_{x : p_X(x) > 0} p_X(x) \cdot x.$$

It is the probability-weighted average of values of X.

Examples

Suppose $\mathbb{P}(X = 1) = \frac{1}{2}$, $\mathbb{P}(X = 2) = \frac{1}{4}$, and $\mathbb{P}(X = 3) = \frac{1}{4}$. What is $\mathbb{E}[X]$?

Suppose $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$. What is $\mathbb{E}[X]$?

Examples

What is the expected value of a roll of a 6-sided die?

What is the expected number of heads in 2 coin flips?

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Functions of a random variable

If X is a random variable and $f(x)$ is any function, we can create a new random variable $f(X)$, which maps $s \in S$ to $f(X(s)) \in \mathbb{R}$.

For $Y = f(X)$, we have

$$p_Y(y) = \mathbb{P}(f(X) = y) = \sum_{x: f(x)=y} \mathbb{P}(X = x) = \sum_{x: f(x)=y} p_X(x).$$

Suppose $\mathbb{P}(X = 0) = \frac{1}{2}$ and $\mathbb{P}(X = 1) = \frac{1}{2}$.

- We have $X = X^2$.
- For $Y = (X + 1)^2$, we have $\mathbb{P}(Y = 1) = \frac{1}{2}$ and $\mathbb{P}(Y = 4) = \frac{1}{2}$.

Expectations of functions

The expectation of $Y = f(X)$ is

$$\begin{aligned}
\mathbb{E}[f(X)] &= \sum_{y:p_Y(y)>0} p_Y(y) \cdot y \\
&= \sum_{y:p_Y(y)>0} \sum_{x:f(x)=y} p_X(x) \cdot f(x) \\
&= \sum_{x:p_X(x)>0} p_X(x) \cdot f(x).
\end{aligned}$$

It is the probability-weighted average value of $f(X)$.

Properties of expectation

For constants a, b, what is $\mathbb{E}[aX + b]$?

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$$\begin{aligned}\mathbb{E}[aX + b] &= \sum_{x : p_X(x) > 0} p_X(x) \cdot (ax + b) \\ &= a \sum_{x : p_X(x) > 0} p_X(x) \cdot x + b \sum_{x : p_X(x) > 0} p_X(x) \\ &= a\mathbb{E}[X] + b.\end{aligned}$$

Examples

Let X be the roll of a 6-sided die. What is $\mathbb{E}[X^2]$?

What is the expected number of heads in n coin flips?

Probability as an expected value

We say that I is an indicator variable for the event A if

$$I = \begin{cases} 1 \text{ if } A \text{ occurs} \\ 0 \text{ if } A^c \text{ occurs} \end{cases}$$

Find $\mathbb{E}[I]$

Linearity of expectation

If X, Y are random variables, get new random variable $X + Y$ with

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Also works with n variables $X_1, \ldots, X_n$:

$$\mathbb{E}[a_1 X_1 + \cdots + a_n X_n] = a_1 \mathbb{E}[X_1] + \cdots + a_n \mathbb{E}[X_n].$$

This means that $\mathbb{E}[-]$ is a **linear** function on the space of random variables.

Examples

Suppose we shuffle a deck of n cards. What is the expected number of cards which end up in the same position?

Suppose you put 6 pairs of socks into the laundry, but 3 socks are lost, leaving 9 total socks. What is the expected number of complete pairs?

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Why care about expectations?

Suppose $X_1, \ldots, X_n$ are chosen independently with the same distribution as X. Law of large numbers:

$$\frac{X_1 + \cdots + X_n}{n} \stackrel{n \to \infty}{\to} \mathbb{E}[X].$$

- From 10^6 fair coin flips, probably ≈half of the flips will be heads.
- If this is not true, I should be suspicious the coin is not fair.

Later in the course, we will quantify this.

Decision making under uncertainty

Rational choice theory in economics says:

- Agents have a utility function $u(x)$ depending on the state of the world x.
- Faced with possible random states of the world X_k, agents act to maximize their **expected utility**:

$$\max_k \mathbb{E}[u(X_k)].$$

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Examples:

- Netflix decides whether to invest in a new show.
- Trader decides which stock to buy.

Utility may not depend purely on monetary value.

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