Lecture 12: Prior Estimate

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Abstract

Prior estimate for multiple stochastic integrals.

Recall **Doob Inequality**: A martingale $X = \{X_t, t \geq 0\}$ with finite p th- moment ($p > 1$) satisfies,

$$E\left(\sup_{0\leq s\leq t} |X_s|^p\right) \leq \left(\frac{p}{p-1}\right)^p E\left(|X_t|^p\right).$$

Cauchy-Schwartz inequality,

$$\left|\int f(x)g(x)\,dx\right|^2 \leq \int |f(x)|^2\,dx \int |g(x)|^2\,dx.$$

1 Moments of Multiple Stochastic Integrals

First Moments Lemma: Let $\alpha \in \mathcal{M}\backslash\{v\}$ with $l(\alpha) \neq n(\alpha)$, let $f \in \mathcal{H}_\alpha$ and let ρ and τ be two stopping times with $t_0 \leq \rho \leq \tau \leq T < \infty$, w.p.1. Then

$$E\left(I_\alpha[f(\cdot)]_{\rho,\tau} \mid \mathcal{A}_\rho\right) = 0, \quad w.p.1 \tag{1.1}$$

A Mean-Square Lemma: Let $\rho \leq \tau \leq \rho + \delta \leq T$, then:

$$\begin{aligned} E\left(\sup_{s\in[\rho,\tau]} |I_\alpha[g]_{\rho,s}|^2 | A_\rho\right) &\leq 4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1}\int_\rho^\tau R_{\rho,s}\,ds \\ R_{\rho,s} &= E(\sup_{\rho\leq t\leq s}|g(t)|^2|A_\rho) < \infty. \end{aligned} \tag{1.2}$$

Proof: Induction on α. First $\alpha = (0)$, $l(\alpha) = 1$, $n(\alpha) = 1$.

$$\begin{aligned} &E\left(\sup_{s\in[\rho,\tau]} |\int_\rho^s g(z)dz|^2 \middle| A_\rho\right) \leq E\left(\delta\int_\rho^s |g(z)|^2 dz\middle| A_\rho\right) \\ &\leq 4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1}\int_\rho^\tau R_{\rho,s}\,ds. \end{aligned} \tag{1.3}$$

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$\alpha = (1)$, $l(\alpha) = 1$, $n(\alpha) = 0$:

$$
\begin{aligned}
& E\left(\sup_{s\in[\rho,\tau]} |\int_\rho^s g(z)dW_z|^2 \Big| A_\rho\right) \\
\leq\ & 4E\left(|\int_\rho^\tau g(z)dW_z|^2 \Big| A_\rho\right) \\
\leq\ & 4\int_\rho^\tau E(|g(z)|^2|A_\rho)dz \\
\leq\ & 4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1}\int_\rho^\tau R_{\rho,s}\,ds. \qquad (1.4)
\end{aligned}
$$

The factor 4 comes from Doob's inequality for martingales.
$\alpha = (\alpha_1, \cdots, \alpha_{k+1})$, $l(\alpha) = k+1$.

Case I: $\alpha_{k+1} = 0$.

$$
\begin{aligned}
& E\left(\sup_{s\in[\rho,\tau]} |\int_\rho^s I_{\alpha-}[g]_{\rho,z}dz|^2 \Big| A_\rho\right) \\
\leq\ & E\left(\sup_{s\in[\rho,\tau]} (s-\rho)\int_\rho^s |I_{\alpha-}[g]_{\rho,z}|^2dz \Big| A_\rho\right) \\
\leq\ & E\left(\delta\int_\rho^\tau |I_{\alpha-}[g]_{\rho,z}|^2dz \Big| A_\rho\right) \\
\leq\ & \delta^2 E\left(\sup_{s\in[\rho,\tau]} |I_{\alpha-}[g]_{\rho,s}|^2 \Big| A_\rho\right), \qquad (1.5)
\end{aligned}
$$

by induction:

$$
\begin{aligned}
\leq\ & \delta^2\delta^{l(\alpha-)+n(\alpha-)-1}4^{l(\alpha-)-n(\alpha-)}\int_\rho^\tau R_{\rho,z}\,dz \\
\leq\ & 4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1}\int_\rho^\tau R_{\rho,z}\,dz. \qquad (1.6)
\end{aligned}
$$

Case II: $\alpha_{k+1} \neq 0$. By Doob, induction:

$$\begin{aligned}
& E\left(\sup_{s\in[\rho,\tau]} |\int_\rho^s I_{\alpha-}[g]_{\rho,z} dW_z|^2 \Big| A_\rho\right) \\
\leq\ & 4 \sup_{s\in[\rho,\tau]} E\left(|\int_\rho^s I_{\alpha-}[g]_{\rho,z} dW_z|^2 \Big| A_\rho\right) \\
\leq\ & 4 \sup_{s\in[\rho,\tau]} \int_\rho^s E\left(|I_{\alpha-}[g]_{\rho,z}|^2 \Big| A_\rho\right) dz \\
\leq\ & 4\delta E\left(\sup_{s\in[\rho,\tau]} |I_{\alpha-}[g]_{\rho,s}|^2 \Big| A_\rho\right) \\
\leq\ & 4\delta 4^{l(\alpha-)-n(\alpha-)} \delta^{l(\alpha-)+n(\alpha-)-1} \int_\rho^\tau R_{\rho,z}\, dz \\
\leq\ & 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_\rho^\tau R_{\rho,z}\, dz. \qquad (1.7)
\end{aligned}$$

Estimates of Higher Moments (Rough): With the same setting,

$$\left(E\left(|I_\alpha[g(\cdot)]_{\rho,\tau}|^{2q} \mid \mathcal{A}_\rho\right)\right)^{1/q} \leq \left(2(2q-1)e^T\right)^{l(\alpha)-n(\alpha)} (\tau-\rho)^{l(\alpha)+n(\alpha)} R \qquad (1.8)$$

where

$$R = \left(E\left(\sup_{\rho\leq s\leq\tau} |g(s)|^{2q} \mid \mathcal{A}_\rho\right)\right)^{1/q} \qquad (1.9)$$

2 Estimate of a Multiple Ito Integral

2.1 The estimate

Let $\alpha = (\alpha_1, \alpha_2, \cdots) \neq v$, v the empty index, δ the time step of discretization over $[0,T]$, τ_n's the uniform discrete time steps, g a right continuous adapted process. Let:

$$R_{0,u} = E(\sup_{s\in[0,u]} |g(s)|^2 \Big| A_0) < \infty, \qquad (2.10)$$

$$F_t^\alpha = E(\sup_{z\in[0,t]} |\sum_{n=0}^{n_z-1} I_\alpha[g(\cdot)]_{\tau_n,\tau_{n+1}} + I_\alpha[g(\cdot)]_{\tau_{n_z},z}|^2 \Big| A_0). \qquad (2.11)$$

Then,

$$F_t^\alpha \leq t\delta^{2(l(\alpha)-1)} \int_0^t R_{0,u}\, du, \quad \text{if } \ l(\alpha) = n(\alpha), \qquad (2.12)$$

and

$$F_t^\alpha \leq 4^{l(\alpha)-n(\alpha)+2} \delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0,u}\, du, \quad l(\alpha) \neq n(\alpha), \qquad (2.13)$$

where

$$n_z := \max\{n \in N | \tau_n \le z\}. \tag{2.14}$$

2.2 Remark

The case $l = n$ is the deterministic Riemann integrals, we see that the total error is $O(\delta^{l-1})$ while local error of each term is $O(\delta^l)$. When $l \neq n$, total error is $O(\delta^{\frac{l+n-1}{2}})$. Each term conditioned locally is $O(\delta^{(l+n)/2}) = O(\delta^{\frac{n'}{2}+n})$. So (2.12)-(2.13) derived the "rule of thumb":

1. a deterministic term, e.g. $I_{(0,0)}$, in the truncation error, leads to a global error of size $O(t^{-1}I_{(0,0)})$, or truncation error divided by t (t equal to the step size);
2. a stochastic term, e.g. $I_{(1,0)}$, in the truncation error, leads to a global error of size $O(t^{-1/2}I_{(1,0)})$, or truncation error divided by $t^{1/2}$ (t equal to the step size). See cancellation between (2.20) and (2.21).

Let A_γ be the indices for discretizations of order γ in the truncated Ito-Taylor expansion.

$$\begin{aligned} A_{1/2} &= \{v, (0), (1)\}, \\ A_1 &= \{v, (0), (1), (1,1)\}, \\ A_{1.5} &= \{v, (0), (1), (1,1), (0,1), (1,0), (0,0), (1,1,1)\}, \\ A_2 &= A_{1.5} \cup \{(0,1,1), (1,0,1), (1,1,0), (1,1,1,1)\}. \end{aligned} \tag{2.15}$$

A general expression for A_γ is:

$$A_\gamma = \{l(\alpha) + n(\alpha) \le 2\gamma, \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}. \tag{2.16}$$

2.3 Proof

When $l(\alpha) = n(\alpha)$,

$$\begin{aligned} F_t^\alpha &= E(\sup_{z\in[0,t]} |\int_0^z I_{\alpha-}[g(\cdot)]_{\tau_{n_u},u}\, du|^2 \Big| A_0) \\ &\le t \cdot E(\sup_{z\in[0,t]} \int_0^z |I_{\alpha-}[g(\cdot)]_{\tau_{n_u},u}|^2\, du \Big| A_0) \\ &\le t \int_0^t E(E(\sup_{s\in[\tau_{n_u},u]} |I_{\alpha-}[g(\cdot)]_{\tau_{n_u},s}|^2 \Big| A_{\tau_{n_u}}) \Big| A_0)\, du. \end{aligned} \tag{2.17}$$

By Lemma in (1.2):

$$\begin{aligned} F_t^\alpha &\le t4^{l(\alpha-)-n(\alpha-)}\delta^{l(\alpha-)+n(\alpha-)-1} \int_0^t E(\int_{\tau_{n_u}}^u R_{\tau_{n_u},s}\, ds \Big| A_0)\, du \\ &\le t\delta^{l(\alpha-)+n(\alpha-)} \int_0^t E(R_{\tau_{n_u},u} \Big| A_0) du \\ &\le t\delta^{2(l(\alpha)-1)} \int_0^t R_{0,u} du. \end{aligned} \tag{2.18}$$

When $l(\alpha) \neq n(\alpha)$:

Case I: $n(\alpha-) = n(\alpha) - 1$**.**

$$F_t^\alpha \leq 2E(\sup_{z\in[0,t]} |\sum_{n=0}^{n_z-1} I_\alpha[g]_{\tau_n,\tau_{n+1}}|^2 \Big| A_0) + 2E(\sup_{z\in[0,t]} |I_\alpha[g]_{\tau_{n_z},z}|^2 \Big| A_0). \quad (2.19)$$

For the first term, use Doob inequality and Lemma in (1.2):

$$\begin{aligned}
& E(\sup_{z\in[0,t]} |\sum_{n=0}^{n_z-1} I_\alpha[g]_{\tau_n,\tau_{n+1}}|^2 \Big| A_0) \\
\leq\ & \sup_{z\in[0,t]} 4E[|\sum_{n=0}^{n_z-1} I_\alpha[g]_{\tau_n,\tau_{n+1}}|^2 \Big| A_0) \\
\leq\ & \sup_{z\in[0,t]} 4E(|\sum_{n=0}^{n_z-2} I_\alpha[g]_{\tau_n,\tau_{n+1}}|^2 \\
& +2\sum_{n=0}^{n_z-2} I_\alpha[g]_{\tau_n,\tau_{n+1}} \cdot E[I_\alpha[g]_{\tau_{n_z-1},\tau_{n_z}} \Big| A_{\tau_{n_z-1}}] \\
& +E[|I_\alpha[g]_{\tau_{n_z-1},\tau_{n_z}}|^2 \Big| A_{\tau_{n_z-1}}] \Big| A_0) \quad (2.20) \\
\leq\ & \sup_{z\in[0,t]} 4E(|\sum_{n=0}^{n_z-2} I_\alpha[g]_{\tau_n,\tau_{n+1}}|^2 + \\
& 4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1} \int_{\tau_{n_z-1}}^{\tau_{n_z}} R_{\tau_{n_z-1},u} du \Big| A_0). \quad (2.21)
\end{aligned}$$

Iterating (2.21):

$$\begin{aligned}
\leq\ & \sup_{z\in[0,t]} 4E(|\sum_{n=0}^{n_z-3} I_\alpha[g]_{\tau_n,\tau_{n+1}}|^2 + \\
& 4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1} \int_{\tau_{n_z-2}}^{\tau_{n_z-1}} R_{\tau_{n_z-2},u} du \\
+\ & 4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1} \int_{\tau_{n_z-1}}^{z} R_{\tau_{n_z-1},u} du \Big| A_0) \\
\leq\ & \sup_{z\in[0,t]} 4E(4^{l(\alpha)-n(\alpha)}\delta^{l(\alpha)+n(\alpha)-1} \int_0^z R_{0,u} du | A_0) \\
\leq\ & 4^{l(\alpha)-n(\alpha)+1}\delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0,u} du. \quad (2.22)
\end{aligned}$$

The 2nd term of (2.19) is bounded as:

$$
\begin{aligned}
& E[\sup_{z\in[0,t]} |I_\alpha[g]_{\tau_{n_z},z}|^2 \Big| A_0) \\
= \; & E(\sup_{z\in[0,t]} |\int_{\tau_{n_z}}^z I_{\alpha-}[g]_{\tau_{n_z},u} du|^2 \Big| A_0) \\
\le \; & E(\sup_{z\in[0,t]} (z-\tau_{n_z}) \int_{\tau_{n_z}}^z |I_{\alpha-}[g]_{\tau_{n_z},u}|^2 du \Big| A_0) \\
\le \; & \delta \int_0^t E(E(\sup_{s\in[\tau_{n_u},u]} |I_{\alpha-}[g]_{\tau_{n_u},s}|^2 \Big| A_{\tau_{n_u}}) \Big| A_0) du \\
\le \; & \delta 4^{l(\alpha-)-n(\alpha-)} \int_0^t E(\int_{\tau_{n_u}}^u R_{\tau_{n_u},s}\, ds\, \delta^{l(\alpha-)+n(\alpha-)-1} \Big| A_0) du \\
\le \; & 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0,u} du. \qquad (2.23)
\end{aligned}
$$

Note in the estimate we only need $4^{l(\alpha)-n(\alpha)+2}$.

Case II: $l(\alpha) \neq n(\alpha)$**,** $n(\alpha-) = n(\alpha)$**.**

$$
\begin{aligned}
F_t^\alpha \; = \; & E(\sup_{z\in[0,t]} |\int_0^z I_{\alpha-}[g]_{\tau_{n_u},u} dW_u|^2 | A_0) \\
\le \; & 4 \sup_{z\in[0,t]} E[|\int_0^z I_{\alpha-}[g]_{\tau_{n_u},u} dW_u|^2 \Big| A_0) \\
\le \; & 4 \sup_{z\in[0,t]} \int_0^z E(E|I_{\alpha-}[g]_{\tau_{n_u},u}|^2 \Big| A_{\tau_{n_u}}) \Big| A_0) du \\
\le \; & 4 \int_0^t E(E(\sup_{s\in[\tau_{n_u},u]} |I_{\alpha-}[g]_{\tau_{n_u},s}|^2 | A_{\tau_{n_u}}) \Big| A_0) du \\
\le \; & 4\, 4^{l(\alpha-)-n(\alpha-)} \int_0^t E(\int_{\tau_{n_u}}^u R_{\tau_{n_u},s}\, ds\, \delta^{l(\alpha-)+n(\alpha-)-1} \Big| A_0) du \\
\le \; & 4^{l(\alpha)-n(\alpha)} \delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0,u} du. \qquad (2.24)
\end{aligned}
$$

Proof is complete.