Lecture 14: Exponential Distribution

Statistics 251

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Lecture Outline

Exponential random variables

Minimum of independent exponentials

Memoryless property

Examples

Where are we?

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Examples

Definition: $X \sim \exp(\lambda)$

Say X is an **exponential random variable of parameter** λ when its probability distribution function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

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So that $\mathbb{P}(X < a) = 1 - e^{-\lambda a}$, what is $\mathbb{P}(X > a)$?

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So $\mathbb{E}X = \frac{1}{\lambda}$ $\operatorname{Var} X = \frac{1}{\lambda^2}$.

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Minimum of independent exponentials is exponential

If $X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)$, and X_1 and X_2 are independent, then $\min\{X_1, X_2\} \sim \exp(\lambda_1 + \lambda_2)$

Proof:

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Proof: Consider $\mathbb{P}(X_1 > a) = e^{\lambda_1 a}$ given $a \geq 0$,

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$$
\begin{aligned}
&P(Y > b|X > a) \\
=&\mathbb{P}(X - a > b|X > a) \\
=&
\end{aligned}
$$

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Example: Roadtrip

Suppose that the number of miles that a car can run before some part wears out is exponentially distributed with an average value of $10,000$ miles.
If the odometer shows the car has already run for $5,000$ miles.
What is the probability that he or she will be able to complete a $5,000$ miles trip without having to replace that part?

Example: Radioactive Decay

Suppose you start at time zero with n radioactive particles. Suppose that each one (independently of the others) will decay at a random time, which follows $\exp(\lambda)$. Let T be amount of time until no particles are left. What are $\mathbb{E}[T]$ and $\text{Var}[T]$?

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Then what is distribution of T_i?

So $\mathbb{E}[T] = \lambda^{-1}(1 + 1/2 + 1/3 + \cdots 1/n)$,
$\text{Var}[T] = \lambda^{-2}(1 + 1/2^2 + 1/3^2 + \cdots 1/n^2)$