Math 18500 - Problem Set 8

Problem 1. Find the following Laplace transforms and inverse Laplace transforms. Do not give answers involving complex numbers.

a. $\mathcal{L}\{f(t)\}$, where $f(t) = e^{-t} + \cos(2t) + 3t^4e^{5t} + 6te^{-7t}\sin(8t) - 9\delta(10-t)$.

b. $\mathcal{L}\{g(t)\}$, where $g(t)$ is defined by

$$g(t) = \begin{cases} 1 & t \le 1 \\ 2t-1 & 1 \le t \le 2 \\ 5-t & 2 \le t \le 3 \\ 2 & t \ge 3 \end{cases}$$

Also sketch the graph of $g(t)$ - you will see that it is a continuous function.

c. $\mathcal{L}^{-1}\{F(s)\}$, where $F(s) = \dfrac{1+s^2}{8-s^3}$.

d. $\mathcal{L}^{-1}\{G(s)\}$, where $G(s) = \dfrac{1}{(1-s^2)^2}$.

Problem 2. Solve the following system of first order equations:

$$\begin{cases} \dfrac{dx}{dt} = 2x + y + e^{-t}\ , & x(0) = 1 \\ \dfrac{dy}{dt} = 2x + 3y - e^{t}\ , & y(0) = 2 \end{cases}$$

Note: This system cannot be solved using the standard methods from week 3, because it is not autonomous. To solve it, take the Laplace transforms of both equations and solve the resulting system of equations for $X(s)$ and $Y(s)$. Then apply the inverse Laplace transform to obtain $x(t)$. To obtain $y(t)$, you can either take the inverse Laplace transform of $Y(s)$, or solve the first equation for $y(t)$ in terms of $x(t)$.

Problem 3. Consider a damped oscillator which is modelled by the equation

$$y'' + 3y' + 2y = f(t)$$

a. Find the transfer function of the oscillator, i.e. the ratio of Laplace transforms $H(s) = \frac{Y(s)}{F(s)}$, where $y(t)$ satisfies the initial conditions $y(0^-) = y'(0^-) = 0$.

b. Find the impulse response of the oscillator, i.e. the solution of the initial value problem

$$y'' + 3y' + 2y = \delta(t)\ ,\ \ y(0^-) = y'(0^-) = 0$$

c. Solve the initial value problem

$$y'' + 3y' + 2y = \frac{1}{1+e^t}\ ,\ \ y(0) = 0\ ,\ \ y'(0) = 0$$

by convolving the forcing term with the impulse response function.

d. Let r be an unspecified constant. Solve the initial value problem

$$y'' + 3y' + 2y = e^{rt}\ ,\ \ y(0) = 1\ ,\ \ y'(0) = 3,$$

by taking the Laplace transform of both sides of the equation. Verify that the solution is of the form

$$y = H(r)e^{rt} + Ae^{-t} + Be^{-2t}$$

where $H(s)$ is the transfer function and A and B are constants (depending of r).

e. Let ω be an unspecified constant. Use part **d** to show that the initial value problem

$$y'' + 3y' + 2y = \cos(\omega t)\ ,\ \ y(0) = 1\ ,\ \ y'(0) = 3$$

has a solution of the form

$$y = |H(i\omega)|\cos(\omega t - \phi) + Ce^{-t} + De^{-2t}.$$

where C, D, and ϕ are constants (all depending on ω). Therefore, $|H(i\omega)|$ is the frequency response function of the oscillator.

Problem 4. The current $j(t)$ in a circuit with a resistor, capacitor, and inductor is modelled by an equation

$$\frac{d^2 j}{dt^2} + 7\frac{dj}{dt} + 12j = \frac{dv}{dt}$$

where $v(t)$ is the voltage supplied to the circuit. The circuit is disconnected until time $t = 0$, so

$$j(t) = v(t) = 0 \text{ for } t < 0$$

and

$$j(0^-) = j'(0^-) = v(0^-) = 0$$

a. Let $J(s)$ and $V(s)$ be the Laplace transforms of $j(t)$ and $v(t)$. Find the ratio

$$H(s) = \frac{J(s)}{V(s)}.$$

In this context one refers to $H(s)$ as the transfer function of the circuit, because it is the ratio of the Laplace transform of the response (j) to the Laplace transform of the input (v).

b. Find the frequency response function of the oscillator, i.e. the amplitude of its steady state response to an alternating voltage $v(t) = \sin(\omega t)$ like the one provided by a wall outlet.

c. Suppose that at time $t = 0$ the circuit is connected to a 1-volt battery. Then

$$v(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Use the result in part **a** to solve for j, and plot it as a function of t. At what time does $j(t)$ attain its maximum value? Label it in your plot.

*Note: Technically the derivative of $v(t)$ is a delta function - it has an infinite value at $t = 0$. Using the result from part **a** allows you to avoid thinking about this delta function explicitly.*

d. Suppose the circuit is connected to the battery at time $t = 0$ and it is disconnected one second later. Then

$$v(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

Use the result in part **a** to solve for j. Give formulas for $j(t)$ on the intervals $0 < t < 1$ and $t > 1$, and plot j as a function of t. At what time does $j(t)$ attain its minimum value?

Problem 5. [*Optional*] Consider the swing from problem 5 of problem set 6, which was modelled by

$$30\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 25\theta = f(t)$$

Suppose that the swing is at rest for $t < 0$, so that

$$\theta(0) = \theta'(0) = 0.$$

a. Find the impulse response function, transfer function, and frequency response function for the swing.

b. You give the swing a push every T seconds. This means $f(t)$ can be modeled by an "impulse train",

$$f(t) = J\sum_{n=0}^{\infty}\delta(t - nT),$$

which can be viewed as the limit as $\Delta t \to 0$ of the forcing term considered in problem set 6, with

$$J = F\Delta t.$$

Given this forcing term, show that the Laplace transform of the solution is

$$\Theta(s) = \frac{JH(s)}{1 - e^{-sT}}$$

where $H(s)$ is the transfer function of the swing. *Hint: You will need to sum a geometric series.*

c. We know that the equation has a steady-state solution $\theta_p(t) = g(t)$ which is a T-periodic function. In general, show that the Laplace transform of any T-periodic function $g(t)$ is given by

$$G(s) = \frac{TG_0(s)}{1 - e^{-sT}}$$

where

$$G_0(s) = \frac{1}{T}\int_0^T g(t)e^{-st}dt.$$

Notice that the Fourier coefficients of $g(t)$ can be recovered from $G_0(s)$ by substituting $s = in\omega$:

$$G_0(in\omega) = \frac{1}{T}\int_0^T g(t)e^{-in\omega t}dt\ , \quad \omega = \frac{2\pi}{T}$$

Hint: Write out the definition of $G(s)$, and split the integral up into a sum of integrals over the intervals $[nT, (n+1)T]$. Make changes of variables to convert each of these integrals to an integral over the interval $[0, T]$. Pull out factors of e^{-snT}, and sum the resulting geometric series.

d. We know that the solution $\theta(t)$ of the initial value problem can be written in the form

$$\theta(t) = \theta_p(t) + \theta_h(t)$$

where the particular solution $\theta_p(t) = g(t)$ is a T-periodic function (the steady state response) and $\theta_h(t)$ is a solution of the homogeneous equation (the transient). By taking the Laplace transform of the equation above, show that

$$\frac{JH(s)}{1 - e^{-sT}} = \frac{TG_0(s)}{1 - e^{-sT}} + (A + Bs)H(s)$$

for some constants A and B. *Note: All you have to do here is find the Laplace transform of $\theta_h(t)$.*

e. Now multiply by $1 - e^{sT}$, set $s = in\omega$, and solve for the Fourier coefficients of $\theta_p(t)$. Verify that the result is correct by comparing it to the one you obtained in problem set 6.

Problem 6. [*Optional*] For some reason, you are trying to model an outbreak of an infectious viral disease. In your model, the infectiousness of the disease is given by a function $n(t)$. What this means is that if a "primary infection" occurs at time $t = 0$, then

$$\int_{t_0}^{t_1} n(t)dt.$$

is the number of resulting "secondary infections" which are expected to occur between times t_0 and t_1, as a result of the primary infected person going around and infecting other people. As a special case, the

reproductive number of the virus (which is defined to be the number of secondary infections that result from a given primary infection), can be calculated in terms of $n(t)$ as follows:

$$R = \int_0^\infty n(t)dt.$$

Now let $b(t)$ be the rate at which people are being infected by the virus (new infections per unit of time). Your model implies that $b(t)$ satisfies the **Euler-Lotka equation**

$$b(t) = \int_0^\infty b(t-u)n(u)du.$$

To see why, observe that the quantity $b(t-u)du$ represents the number of people who became infected between times $t-u-du$ and $t-u$. Multiplying this by the infectiousness $n(u)$ gives the rate at which those people are infecting new people at time t, and summing over all values of u gives the total rate of new infections at time t.

In real-world scenarios it can be difficult to estimate the function $n(t)$. It is easier to estimate the function

$$g(t) = \frac{n(t)}{R}$$

which is normalized so that

$$\int_0^\infty g(t)dt = \frac{1}{R}\int_0^\infty n(t)dt = 1.$$

This function $g(t)$ is called the **generation interval distribution**, and it represents the probability that the time gap between a primary infection and a resulting secondary infections is equal to t. More precisely,

$$\int_{t_0}^{t_1} g(t)dt = \frac{1}{R}\int_{t_0}^{t_1} n(t)dt$$

is the fraction of primary-secondary pairs such that the gap is more than t_0 and less than t_1.

a. Assume that the epidemic is growing (or dying off) exponentially, so that

$$b(t) = e^{rt}$$

for some constant r (if $r > 0$, then the epidemic is growing, and if $r < 0$, the epidemic is dying off). Use the Euler-Lotka equation, together with the definitions of R and $g(t)$, to show that

$$R = \frac{1}{G(r)}$$

where

$$G(s) = \int_0^\infty g(t)e^{-st}dt$$

is the Laplace transform of $g(t)$.

b. Show from the definitions of R, $b(t)$, $n(t)$, and $g(t)$ that

$$G(0) = 1.$$

This means that if the epidemic is stable ($r = 0$) then each infected person is infecting one new person on average ($R = 1$).

c. In general, show that R is an increasing function of r. It follows that $R > 1$ for an epidemic which is growing, and $R < 1$ for an epidemic which is dying off.
Hint: By definition, the function $g(t)$ is positive - use this to show that the derivative $G'(r)$ is the integral of a negative function, and therefore $G(r)$ is a decreasing function of r.

d. Suppose (unrealistically) that the gap between primary and secondary infections is always a specific time T. This corresponds to a generation interval distribution

$$g(t) = \delta(t-T)$$

Solve for R in terms of r, in this case.

e. More realistically, suppose that the generation interval distribution is a **gamma distribution**

$$g(t) = \gamma\, t^{k-1} e^{-\beta t},$$

where k and β are constants, and γ is a normalization factor chosen so that

$$\int_0^\infty g(t)dt = \int_0^\infty \gamma\, t^{k-1} e^{-\beta t} dt = 1.$$

You may assume that k is a non-negative integer. Show that $\gamma = \frac{\beta^k}{k!}$ (*Hint:* $G(0) = 1$) and that

$$R = (1 + r/\beta)^k$$

Note: This relationship between R and r is valid even if k is not an integer. But to give a formula for γ in this case, we must introduce the Γ function,

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

This function is very interesting because it extends the factorial function to arbitrary real values! In particular, you can show that

$$\Gamma(k+1) = k!$$

for any integer k. For positive real values of k, the correct formula for γ turns out to be

$$\gamma = \frac{\beta^k}{\Gamma(k+1)}.$$

f. In practice, the function $g(t)$ can be estimated by taking a random sample of primary-secondary pairs and making a histogram of the gaps between the date of the primary infection and the date of the resulting secondary infection. See the following paper (on which this problem was based):

https://www.ncbi.nlm.nih.gov/pmc/articles/PMC1766383/pdf/rspb20063754.pdf

for an example of such a histogram (on page 602). It is reasonable to approximate $g(t)$ by thinking of the histogram as the graph of a piecewise function,

$$g(t) = p_d\ , \quad d < t < d+1,$$

where p_d is the fraction of observed primary-secondary pairs such that the gap between the infections is d days. Assuming that $g(t)$ takes this form, show that

$$\frac{1}{R} = \frac{e^r - 1}{r} \sum_{d=0}^{\infty} p_d e^{-rd}$$

This formula can be used to estimate the reproductive number of an emerging infectious disease, assuming that the disease is observed to be spreading exponentially, and sufficient data has been collected about primary-secondary pairs of infections.