

Lecture 8: Ito-Taylor Expansion I

Zhongjian Wang*

Abstract

Derivation of Ito-Taylor Expansion, notation of stochastic integrals, coefficient functions

1 Deterministic Analogue

Consider ODE:

$$\frac{d}{dt}X = a(X), \quad X(t) = X_0 + \int_0^t a(X(s)) ds, \quad (1.1)$$

chain rule on any f :

$$\frac{d}{dt}f(X) = a(X)f'(X) \equiv Lf(X), \quad (1.2)$$

or:

$$f(X) = f(X_0) + \int_0^t Lf(X)(s) ds. \quad (1.3)$$

Applying (1.3) to $a(X(s))$ in (1.1), we get:

$$\begin{aligned} X(t) &= X_0 + \int_0^t \left(a(X_0) + \int_0^s La(X)(z) dz \right) ds \\ &= X_0 + a(X_0) \int_0^t ds + \int_0^t \int_0^s La(X)(z) dz ds. \end{aligned} \quad (1.4)$$

Repeating once more:

$$X(t) = X_0 + a(X_0) \int_0^t ds + La(X_0) \int_0^t \int_0^s dz ds + R, \quad (1.5)$$

with:

$$R = \int_0^t \int_0^{s_3} \int_0^{s_2} L^2 a(X)(s_1) ds_1 ds_2 ds_3.$$

*Department of Statistics, University of Chicago

Continuing this process n times, one recovers Taylor expansion in integral form:

$$\begin{aligned}
f(X)(t) &= f(X_0) + \sum_{j=1}^n \frac{t^j}{j!} (L^j f)(X_0) \\
&\quad + \int_0^t \int_0^{s_{n+1}} \cdots \int_0^{s_2} (L^{n+1} f)(X)(s_1) ds_1 \cdots ds_{n+1}.
\end{aligned} \tag{1.6}$$

2 Ito-Taylor Expansion

Consider Ito eqn:

$$X_t = X_0 + \int_0^t a(X_s) ds + \int_0^t b(X_s) dW_s, \tag{2.7}$$

Ito formula gives:

$$f(X_t) = f(X_0) + \int_0^t (L^0 f)(X_s) ds + \int_0^t (L^1 f)(X_s) dW_s, \tag{2.8}$$

$$L^0 = a \frac{d}{dx} + \frac{1}{2} b^2 \frac{d^2}{dx^2},$$

$$L^1 = b \frac{d}{dx}.$$

Substitute (2.8) into (2.7):

$$\begin{aligned}
X_t &= X_0 \\
&\quad + \int_0^t \left(a(X_0) + \int_0^s (L^0 a)(X_z) dz + \int_0^s (L^1 a)(X_z) dW_z \right) ds \\
&\quad + \int_0^t \left(b(X_0) + \int_0^s (L^0 b)(X_z) dz + \int_0^s (L^1 b)(X_z) dW_z \right) dW_s \\
&= X_0 + a(X_0) \int_0^t ds + b(X_0) \int_0^t dW_s + R,
\end{aligned} \tag{2.9}$$

remainder:

$$\begin{aligned}
R &= \int_0^t \int_0^s (L^0 a)(X_z) dz ds + \int_0^t \int_0^s (L^1 a)(X_z) dW_z ds \\
&\quad + \int_0^t \int_0^s (L^0 b)(X_z) dz dW_s + \int_0^t \int_0^s (L^1 b)(X_z) dW_z dW_s.
\end{aligned} \tag{2.10}$$

Continuing once more by applying Ito formula to $L^1 b(X_z)$:

$$\begin{aligned}
X_t &= X_0 + a(X_0) \int_0^t ds + b(X_0) \int_0^t dW_s \\
&\quad + (L^1 b)(X_0) \int_0^t \int_0^s dW_z dW_s + R_1,
\end{aligned} \tag{2.11}$$

remainder:

$$\begin{aligned}
R_1 &= \int_0^t \int_0^s (L^0 a)(X_z) dz ds + \int_0^t \int_0^s (L^1 a)(X_z) dW_z ds \\
&+ \int_0^t \int_0^s (L^0 b)(X_z) dz dW_s + \int_0^t \int_0^s \int_0^z (L^0 L^1 b)(X_u) du dW_z dW_s \\
&+ \int_0^t \int_0^s \int_0^z (L^1 L^1 b)(X_u) dW_u dW_z dW_s.
\end{aligned} \tag{2.12}$$

This is called Ito-Taylor formula, in principle, one could continue given enough smoothness of a and b , to generate an expansion. The remainder involves multiple stochastic Ito integrals.

3 Shortened Notations

3.1 Multi-indices

We shall call a row vector

$$\alpha = (j_1, j_2, \dots, j_l)$$

where

$$j_i \in \{0, 1, \dots, m\}$$

for $i \in \{1, 2, \dots, l\}$ and $m = 1, 2, 3, \dots$, a multi-index of length

$$l := l(\alpha) \in \{1, 2, \dots\}$$

Here m will denote the number of components of the Wiener process under consideration. For completeness we denote by v the multi-index of length zero, that is with

$$l(v) := 0$$

Thus, for example,

$$l((1, 0)) = 2 \text{ and } l((1, 0, 1)) = 3.$$

In addition, we shall write $n(\alpha)$ for the number of components of a multi-index which are equal to 0. For example,

$$n((1, 0, 1)) = 1, \quad n((0, 1, 0)) = 2, \quad n((0, 0)) = 2.$$

We denote the set of all multi-indices by \mathcal{M} , so

$$\mathcal{M} = \{ (j_1, j_2, \dots, j_l) : j_i \in \{0, 1, \dots, m\}, i \in \{1, \dots, l\}, \text{ for } l = 1, 2, 3, \dots \} \cup \{v\}. \tag{3.13}$$

Given $\alpha \in \mathcal{M}$ with $l(\alpha) \geq 1$, we write $-\alpha$ and $\alpha-$ for the multi-index in \mathcal{M} obtained by deleting the first and the last component, respectively, of α . Thus

$$\begin{aligned} -(1, 0) &= (0), (1, 0)- = (1) \\ -(0, 1, 1) &= (1, 1), (0, 1, 1)- = (0, 1). \end{aligned}$$

Finally, for any two multi-indices $\alpha = (j_1, j_2, \dots, j_k)$ and $\bar{\alpha} = (\bar{j}_1, \bar{j}_2, \dots, \bar{j}_l)$ we introduce an operation $*$ on \mathcal{M} by

$$\alpha * \bar{\alpha} = (j_1, j_2, \dots, j_k, \bar{j}_1, \bar{j}_2, \dots, \bar{j}_l)$$

the multi-index formed by adjoining the two given multi-indices. We shall call this the concatenation operation. For example, for $\alpha = (0, 1, 2)$ and $\bar{\alpha} = (1, 3)$ we have

$$\alpha * \bar{\alpha} = (0, 1, 2, 1, 3) \text{ and } \bar{\alpha} * \alpha = (1, 3, 0, 1, 2)$$

3.2 Multiple Ito Integrals

Given integrable condition, we can recursively define,

$$I_\alpha[f(\cdot)]_{\rho, \tau} := \begin{cases} f(\tau) & : l = 0 \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho, s} ds & : l \geq 1 \text{ and } j_l = 0 \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho, s} dW_s^{j_l} & : l \geq 1 \text{ and } j_l \geq 1, \end{cases}$$

where $l(\alpha) = l$, for example,

$$\begin{aligned} I_0[f]_{0, t} &= \int_0^t f(s) ds; \\ I_1[f]_{0, t} &= \int_0^t f(s) dW_s; \\ I_{(0,1)}[f]_{0, t} &= \int_0^t \int_0^{s_2} f(s_1) ds_1 dW_{s_2} \end{aligned} \tag{3.14}$$

and $I_j = I_j[1]$. Then Ito-Taylor expansion up to two layer integrals is:

$$\begin{aligned} X_t &= X_0 + aI_0 + bI_1 + (aa' + \frac{1}{2}b^2a'')I_{(0,0)} \\ &\quad + [ab' + \frac{1}{2}b^2b'']I_{(0,1)} + ba'I_{(1,0)} + bb'I_{(1,1)} + \dots, \end{aligned} \tag{3.15}$$

dots mean higher layered integrals.

Some Calculations

- For convenience, we write

$$I_{\alpha,t} = I_{\alpha}[1]_{0,t}, \quad W_t^0 = t.$$

- Let $j_1, \dots, j_l \in \{0, 1, \dots, m\}$ and $\alpha = (j_1, \dots, j_l) \in \mathcal{M}$ where $l = 1, 2, 3, \dots$. Then

$$W_t^j I_{\alpha,t} = \sum_{i=0}^l I_{(j_1, \dots, j_i, j, j_{i+1}, \dots, j_l), t} + \sum_{i=1}^l \mathbf{1}_{\{j_i=j \neq 0\}} I_{(j_1, \dots, j_{i-1}, 0, j_{i+1}, \dots, j_l), t} \quad (3.16)$$

for all $t \geq 0$.

Sketch of Proof: By Ito formula,

$$W_t^j I_{\alpha,t} = I_{(j),t} I_{\alpha,t} = \int_0^t I_{\alpha,s} dI_{(j),s} + \int_0^t I_{(j),s} I_{\alpha-,s} dW_s^{j_i} + \mathbf{1}_{\{j_i=j \neq 0\}} \int_0^t I_{\alpha-,s} ds \quad (3.17)$$

$$= I_{(j_1, \dots, j_l, j), t} + \int_0^l W_s^j I_{\alpha-,s} dW_s^{j_i} + \mathbf{1}_{\{j_i=j \neq 0\}} I_{(j_1, \dots, j_{l-1}, 0), t} \quad (3.18)$$

Now, consider $W_s^j I_{\alpha-,s}$ by induction.

- (Corollary) Suppose that $\alpha = (j_1, \dots, j_l)$ with $j_1 = \dots = j_l = j \in \{0, \dots, m\}$ where $l \geq 2$. Then for $t \geq 0$

$$I_{\alpha,t} = \begin{cases} \frac{1}{l!} t^l & : j = 0 \\ \frac{1}{l} (W_t^j I_{\alpha-,t} - t I_{(\alpha-)-,t}) & : j \geq 1 \end{cases}$$

Sketch of Proof: The case $j = 0$ follows from the usual deterministic integration rule. For $j \in \{1, \dots, m\}$ the relation (3.16) gives

$$t I_{(\alpha-)-,t} = \sum_{i=0}^{l-2} I_{(j_1, \dots, j_i, 0, j_{i+1}, \dots, j_{l-2}), t} \text{ and}$$

$$W_t^j I_{\alpha-,t} = l I_{\alpha,t} + \sum_{i=1}^{l-1} I_{(j_1, \dots, j_{i-1}, 0, j_{i+1}, \dots, j_{l-1}), t}$$

Examples:

$$I_{(j,j),t} = \frac{1}{2!} (I_{(j),t}^2 - t) \quad (3.19)$$

$$I_{(j,j,j),t} = \frac{1}{3!} (I_{(j),t}^3 - 3t I_{(j),t}) \quad (3.20)$$

$$I_{(j,j,j,j),t} = \frac{1}{4!} (I_{(j),t}^4 - 6t I_{(j),t}^2 + 3t^2) \quad (3.21)$$

4 Coefficient Functions

We shall write the diffusion operator for the Ito equation in d dimension defined with m dimension Wiener process as

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^d a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^d \sum_{j=1}^m b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^l}$$

and for $j \in \{1, \dots, m\}$ introduce the operator

$$L^j = \sum_{k=1}^d b^{k,j} \frac{\partial}{\partial x^k}$$

For each $\alpha = (j_1, \dots, j_l)$ and function f (which has enough regularity), we define recursively the Ito coefficient function

$$f_\alpha = \begin{cases} f & : l = 0 \\ L^{j_1} f_{-\alpha} & : l \geq 1 \end{cases}$$

If the function f is not explicitly stated we shall always take it to be the identity function $f(t, x) \equiv x$. For example, in the 1-dimensional case $d = m = 1$ for $f(t, x) \equiv x$ we have

$$f_{(0)} = a, \quad f_{(1)} = b, \quad f_{(1,1)} = bb'$$

and

$$f_{(0,1)} = ab' + \frac{1}{2}b^2b''$$

Here the prime $'$ denotes the ordinary or partial derivative with respect to the x variable, depending on whether or not the function being differentiated depends only on x or on both t and x .

Now note

$$\begin{aligned} f_{(0)} &= a, & f_{(j_1)} &= b^{j_1} \\ f_{(0,0)} &= aa' + \sigma a'', & f_{(0,j_1)} &= ab^{j_1'} + \sigma b^{j_1''} \\ f_{(j_1,0)} &= b^{j_1} a', & f_{(j_1,j_2)} &= b^{j_1} b^{j_2'} \end{aligned} \tag{4.22}$$

given

$$\sigma = \frac{1}{2} \sum_{j=1}^m (b^j)^2 \tag{4.23}$$