Lecture 18: Cahgne of variables (multivariate)

Statistics 251

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Lecture Outline

Formula

Examples

Where are we?

Formula

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Let X_1 and X_2 be jointly continuous random variables with joint probability density function f_{X_1,X_2}. Suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$

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1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2, with solutions given by, say, $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

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2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the 2×2 determinant, i.e.

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_2}{\partial x_1}\frac{\partial g_1}{\partial x_2} \neq 0$$

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   at all points (x_1, x_2).

Then Y_1 and Y_2 are jointly continuous with joint density function given by

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}$$

where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

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Let $X = (X_1, \cdots, X_n)$ be jointly continuous random variables with joint probability density function f_X. Suppose that $Y = g(X)$

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Then Y are jointly continuous with joint density function given by

$$f_Y(y) = f_X(x)|J(x)|^{-1}$$

where $x = h(y)$.

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Example: 2D transform

Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2} .

Example: general linear transform

Let $X = (X_1, \cdots, X_n)$ be jointly continuous random variables with joint probability density function f_X. Suppose that $Y = AX$ where A is a $n \times n$ matrix with $|A| \neq 0$. Find the joint density function of Y in terms of f_X .

Example: 2D normal r.v. in the polar coodinate

Let (X, Y) denote a random point in the plane, and assume that the rectangular coordinates X and Y are independent standard normal random variables. What is the joint distrubtion of (r, θ), the polar coordinate representation of (x, y)?

Example: 2D normal r.v. in the polar coodinate

Let (X, Y) denote a random point in the plane, and assume that the rectangular coordinates X and Y are independent standard normal random variables. What is the joint distrubtion of (r, θ), the polar coordinate representation of (x, y)?
To be more specific, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} y/x$.

Summation of i.i.d. exp varaibles

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed exponential random variables with rate λ. Let

$$Y_i = X_1 + \cdots + X_i \quad i = 1, \cdots, n$$

1. Find the joint density function of $Y_1, \cdots, Y_n$.
2. Use the first result to find the density of Y_n.