Lecture 24: Strong law of large numbers

Statistics 251

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Lecture Outline

Statement of Theorem

Proof

Examples

Where are we?

Comparison with weak law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, each having a finite mean $\mu = E[X_i]$. Then,

weak for any $\epsilon > 0$,

$$\mathbb{P}\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0 \quad \text{as} \quad n \to \infty$$

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The strong implies the weak.

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- in distribution if
$$\lim_{n\to\infty} \mathbb{P}\left(X_n \in A\right) = \mathbb{P}(X \in A) \quad \text{for cointinuous set } A \text{ in } R$$
e.g. Central limit theorem

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Proof

Although the theorem can be proven without this assumption, we will suppose that $\mathbb{E}\left[X_i^4\right] = K < \infty$.
Let $S_n = \sum_{i=1}^{n} X_i$ and consider

$$\begin{aligned}\mathbb{E}\left[S_n^4\right] =& \mathbb{E}\left[(X_1 + \cdots + X_n)(X_1 + \cdots + X_n)\right.\\ &\left.\times (X_1 + \cdots + X_n)(X_1 + \cdots + X_n)\right]\end{aligned}$$

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$$\begin{aligned}\mathbb{E}\left[S_n^4\right] &= n\mathbb{E}\left[X_i^4\right] + 6\begin{pmatrix} n \\ 2 \end{pmatrix}\mathbb{E}\left[X_i^2 X_j^2\right]\\ &= nK + 3n(n-1)\mathbb{E}\left[X_i^2\right]\mathbb{E}\left[X_j^2\right]\end{aligned}$$

Proof conti

We know

$$0 \leq \text{Var}\left(X_i^2\right) \mathbb{E}\left[X_i^4\right] - \left(\mathbb{E}\left[X_i^2\right]\right)^2$$

So,

$$\left(\mathbb{E}\left[X_i^2\right]\right)^2 \leq \mathbb{E}\left[X_i^4\right] = K.$$

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Hence

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right] = \sum_{n=1}^{\infty} \mathbb{E}\left[\frac{S_n^4}{n^4}\right] < \infty$$

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What if with some positive probability,

$$\sum_{n=1}^{\infty} S_n^4/n^4 \quad \text{diverges?}$$

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Alternative (classic) definition of probability of an event

We suppose that an experiment, whose sample space is S, is repeatedly performed under exactly the same conditions. For each event E of the sample space S, we define $n(E)$ to be the number of times in the first n repetitions of the experiment that the event E occurs. Then $P(E)$, the probability of the event E, is defined as

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Now let

$$X_i = \begin{cases} 1 & \text{if } E \text{ occurs on the } i \text{ th trial} \\ 0 & \text{if } E \text{ does not occur on the } i \text{ th trial} \end{cases}$$

we have, by the strong law of large numbers, that with probability 1 ,

$$\frac{n(E)}{n} = \frac{X_1 + \cdots + X_n}{n} \to E[X] = \mathbb{P}(E)$$

Bernstein Polynomials

Let $f(x)$ be a continuous function defined for $0 \leq x \leq 1$. Consider the functions

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}$$

and prove that

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Hint: Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with mean x. Show that

$$B_n(x) = E\left[f\left(\frac{X_1 + \cdots + X_n}{n}\right)\right]$$