# Lecture 23: Central Limit Theorem

## Statistics 251


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# Lecture Outline

Theorem Statement

Moment Generating Function: Part II

Proof of the theorem

Examples

# Where are we?

# Revision

Recall the weak law of large numbers,
Let $X_1, X_2, \ldots$ be a sequence of thdependent and Identically distributed random vanables, each having finite mean $E[X_i] = \mu$.
Then, for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0 \quad \text{as} \quad n \to \infty$$

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In some sense means, $\left|\frac{X_1+\cdots+X_n-n\mu}{n}\right| \to 0$. What about

$$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}}$$

or in other words, how fast does it converge?

# Central limit theorem

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, each having mean $\mu$ and variance $\sigma^2$. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$,

$$\mathbb{P}\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx \text{ as } n \to \infty$$

# Where are we?

# Revision

Definition:

$$M(t) = E\left[e^{tX}\right]$$
$$= \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete with mass function } p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } X \text{ is continuous with density } f(x) \end{cases}$$

When calculating moments

$$M^n(0) = E\left[X^n\right] \quad n \geq 1$$

# Moment Generating Function of sum of independent random variables

If $X$ and $Y$ are independent and have moment generating functions $M_X(t)$ and $M_Y(t)$, respectively. Then $M_{X+Y}(t)$, is given by

$$\begin{aligned}
M_{X+Y}(t) &= E\left[e^{t(X+Y)}\right] \\
&= E\left[e^{tX}e^{tY}\right] \\
&= E\left[e^{tX}\right]E\left[e^{tY}\right] \\
&= M_X(t)M_Y(t).
\end{aligned}$$

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If $\{X_i\}_{i=1}^n$ are identical independent random variables from the same distribution with moment generating function $M_X$, what is $M_{\sum_{i=1}^n X_i}$?

# Lemma: moment generating function decides the distribution

- Let $Z_1, Z_2, \ldots$ be a sequence of random variables having distribution functions $F_{Z_n}$ and moment generating functions $M_{Z_n}, n \geq 1$.
- Let $Z$ be a random variable having distribution function $F_Z$ and moment generating function $M_Z$.
- If $M_{Z_n}(t) \to M_Z(t)$ for all $t$, then $F_{Z_n}(t) \to F_Z(t)$ for all $t$ at which $F_Z(t)$ is continuous.

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Let $M(t)$ be the moment generating function of $X_i$. Then the moment generating function of $\sum_{i=1}^{n} X_i/\sqrt{n}$ is given by $\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n$.
Let $L(t) = \log M(t)$, then

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# Proof conti

What is the limit of $nL(t/\sqrt{n})$ when $n \to \infty$?

$$
\begin{aligned}
\lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}} &= \lim_{n\to\infty} \frac{-L'(t/\sqrt{n})n^{-3/2}t}{-2n^{-2}} \text{ (by L'Hôpital's rule)} \\
&= \lim_{n\to\infty} \left[\frac{L'(t/\sqrt{n})t}{2n^{-1/2}}\right] \\
&= \lim_{n\to\infty} \left[\frac{-L''(t/\sqrt{n})n^{-3/2}t^2}{-2n^{-3/2}}\right] \text{ (by L'Hôpital's rule )} \\
&= \lim_{n\to\infty} \left[L''\left(\frac{t}{\sqrt{n}}\right)\frac{t^2}{2}\right] \\
&= \frac{t^2}{2} \text{ (as } L''(0) = \frac{M(0)M''(0) - [M'(0)]^2}{[M(0)]^2} = 1)
\end{aligned}
$$

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## Theorem

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- Z score
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| $a$ | -2.58 | -1.96 | -1.645 | 0 | 1.645 | 1.96 | 2.58 |
|---|---|---|---|---|---|---|---|
| $\phi(a)$ | 0.005 | 0.025 | 0.05 | 0.5 | 0.95 | 0.975 | 0.995 |

# Bound for sum of rolling dice

Let $X_i, i = 1, \ldots, 10$, be independent random variables, each uniformly distributed over $(0, 1)$. Calculate an approximation to $P\left\{\sum_{i=1}^{10} X_i > 6\right\}$

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Solution.

First,

$$E\left(X_i\right) = \frac{7}{2}, \quad \operatorname{Var}\left(X_i\right) = E\left[X_i^2\right] - \left(E\left[X_i\right]\right)^2 = \frac{35}{12}$$

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Then

$$\begin{aligned} P\{29.5 \leq X \leq 40.5\} &= P\left\{\frac{29.5-35}{\sqrt{\frac{350}{12}}} \leq \frac{X-35}{\sqrt{\frac{350}{12}}} \leq \frac{40.5-35}{\sqrt{\frac{350}{12}}}\right\} \\ &\approx 2\Phi(1.0184) - 1 \\ &\approx .692 \end{aligned}$$