Lecture 10: Strong Approximation of Stochastic Integrals

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Abstract

Stratonovich Taylor expansion; strong approximation by Fourier series expansion.

1 Stratonovich Taylor Expansion

1.1 1D Case

Given X_t satisfies,

$$X_t = X_{t_0} + \int_{t_0}^{t} a\left(X_s\right) ds + \int_{t_0}^{t} b\left(X_s\right) dW_s, \tag{1.1}$$

we know it also can be re-written as,

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}\left(X_s\right) ds + \int_{t_0}^{t} b\left(X_s\right) \circ dW_s, \tag{1.2}$$

where

$$\underline{a} = a - \frac{1}{2}bb'. \tag{1.3}$$

Given function f,

$$\begin{aligned} f\left(X_t\right) =& f\left(X_{t_0}\right) + \int_{t_0}^{t} (a\frac{\partial}{\partial x} f\left(X_s\right) + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2} f\left(X_s\right))ds + \int_{t_0}^{t} b\frac{\partial}{\partial x} f\left(X_s\right) dW_s \\ =& f\left(X_{t_0}\right) + \int_{t_0}^{t} (a - \frac{1}{2}bb')\frac{\partial}{\partial x} f\left(X_s\right) ds + \int_{t_0}^{t} b\frac{\partial}{\partial x} f\left(X_s\right) \circ dW_s \\ =& f\left(X_{t_0}\right) + \int_{t_0}^{t} \underline{L}^0 f\left(X_s\right) ds + \int_{t_0}^{t} \underline{L}^1 f\left(X_s\right) \circ dW_s \end{aligned} \tag{1.4}$$

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where we note,

$$\underline{L}^0 = \underline{a}\frac{\partial}{\partial x} \tag{1.5}$$

$$\underline{L}^1 = b\frac{\partial}{\partial x}. \tag{1.6}$$

Now,

$$\begin{aligned} X_t =& X_{t_0} \\ &+ \int_{t_0}^{t} \left(\underline{a}\left(X_{t_0}\right) + \int_{t_0}^{s} \underline{L}^0 \underline{a}\left(X_z\right) dz + \int_{t_0}^{s} \underline{L}^1 \underline{a}\left(X_z\right) \circ dW_z \right) ds \\ &+ \int_{t_0}^{t} \left(b\left(X_{t_0}\right) + \int_{t_0}^{s} \underline{L}^0 b\left(X_z\right) dz + \int_{t_0}^{s} \underline{L}^1 b\left(X_z\right) \circ dW_z \right) \circ dW_s \\ =& X_{t_0} + \underline{a}\left(X_{t_0}\right) \int_{t_0}^{t} ds + b\left(X_{t_0}\right) \int_{t_0}^{t} 1 \circ dW_s + R \end{aligned} \tag{1.7}$$

with remainder,

$$\begin{aligned} R =& \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 \underline{a}\left(X_z\right) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1 \underline{a}\left(X_z\right) \circ dW_z ds \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 b\left(X_z\right) dz \circ dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1 b\left(X_z\right) \circ dW_z \circ dW_s \end{aligned}. \tag{1.8}$$

1.2 General Case

Consider,

$$X_t = X_{t_0} + \int_{t_0}^{t} a\left(s, X_s\right) ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j\left(s, X_s\right) dW_s^j \tag{1.9}$$

we have

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}\left(s, X_s\right) ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j\left(s, X_s\right) \circ dW_s^j \tag{1.10}$$

with

$$\underline{a}^i = a^i - \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{d} b^{k,j} \frac{\partial b^{i,j}}{\partial x^k}. \tag{1.11}$$

If

$$\underline{L}^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} \underline{a}^k \frac{\partial}{\partial x^k} \tag{1.12}$$

$$\underline{L}^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k} \tag{1.13}$$

then

$$\underline{a} = a - \frac{1}{2}\sum_{j=1}^{m} \underline{L}^j b^j. \tag{1.14}$$

Multiple stochastic integral and coefficient functions can be defined iteratively,

$$J_\alpha[g(\cdot, X.)]_{\rho,\tau} = \begin{cases} g(\tau, X_\tau) & : \; l = 0 \\ \int_\rho^\tau J_{\alpha-}[g(\cdot, X.)]_{\rho,s} ds & : \; l \geq 1, j_l = 0 \\ \int_\rho^\tau J_{\alpha-}[g(\cdot, X.)]_{\rho,s} \circ dW_s^{j_1} & : \; l \geq 1, j_l \geq 1 \end{cases} \tag{1.15}$$

$$\underline{f}_\alpha = \begin{cases} f & : \; l = 0 \\ \underline{L}^{j_1} \underline{f}_{-\alpha} & : \; l \geq 1 \end{cases} \tag{1.16}$$

Finally, given an hierarchical set $\mathcal{A}$,

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}} J_\alpha \left[\underline{f}_\alpha(\rho, X_\rho)\right]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} J_\alpha \left[\underline{f}_\alpha(\cdot, X.)\right]_{\rho,\tau}. \tag{1.17}$$

Relationship with Ito Integral, given $g \equiv 1$

- For $l(\alpha) \in \{0, 1\}$

$$I_\alpha = J_\alpha. \tag{1.18}$$

- For $l(\alpha) = 2$,

$$I_\alpha = J_\alpha - \frac{1}{2} I_{\{j_1 = j_2 \neq 0\}} J_{(0)}. \tag{1.19}$$

- For $l(\alpha) = 3$,

$$I_\alpha = J_\alpha - \frac{1}{2}\left(I_{\{j_1=j_2\neq 0\}} J_{(0,j_3)} + I_{\{j_2=j_3\neq 0\}} J_{(j_1,0)}\right). \tag{1.20}$$

- For $l(\alpha) = 4$,

$$\begin{aligned} I_\alpha = &J_\alpha + \frac{1}{4} I_{\{j_1=j_2\neq 0\}} I_{\{j_3=j_4\neq 0\}} J_{(0,0)} \\ &- \frac{1}{2}\left(I_{\{j_1=j_2\neq 0\}} J_{(0,j_3,j_4)} + I_{\{j_2=j_3\neq 0\}} J_{(j_1,0,j_4)} + I_{\{j_3=j_4\neq 0\}} J_{(j_1,j_2,0)}\right). \end{aligned} \tag{1.21}$$

Relation between Stratonovich Integrals (can be explained)

$$W_t^j J_{\alpha,t} = \sum_{i=0}^{l} J_{(j_1,\ldots,j_i,j,j_{i+1},\ldots,j_l),t} \tag{1.22}$$

2 Strong Approximation of Stochastic Integrals

Motivation: *Riemann-Stieltjes integrals with respect to such a process will converge to Stratonovich stochastic integrals rather than to Ito stochastic integrals.*

2.1 Differentiable path approximation to a Wiener process

Consider the Brownian bridge process formed from a given m-dimensional Wiener process $W_t = (W_t^1, \ldots, W_t^m)$,

$$\left\{ W_t - \frac{t}{\Delta} W_\Delta, 0 \leq t \leq \Delta \right\}. \tag{2.23}$$

We pathwisely consider Fourier expansion of the process,

$$W_t^j - \frac{t}{\Delta} W_\Delta^j = \frac{1}{2} a_{j,0} + \sum_{r=1}^{\infty} \left(a_{j,r} \cos\left(\frac{2r\pi t}{\Delta}\right) + b_{j,r} \sin\left(\frac{2r\pi t}{\Delta}\right) \right) \tag{2.24}$$

with random coefficients

$$a_{j,r} = \frac{2}{\Delta} \int_0^\Delta \left(W_s^j - \frac{s}{\Delta} W_\Delta^j \right) \cos\left(\frac{2r\pi s}{\Delta}\right) ds, \tag{2.25}$$

$$b_{j,r} = \frac{2}{\Delta} \int_0^\Delta \left(W_s^j - \frac{s}{\Delta} W_\Delta^j \right) \sin\left(\frac{2r\pi s}{\Delta}\right) ds, \tag{2.26}$$

and by setting $t = \Delta$,

$$a_{j,0} = -2 \sum_{r=1}^{\infty} a_{j,r}. \tag{2.27}$$

It can be shown that $a_{j,r}$ and $b_{j,r}$ are $N\left(0; \Delta/(2\pi^2 r^2)\right)$ distributed and pairwise independent.

So we approximate W_t by,

$$W_t^{j,p} = \frac{t}{\Delta} W_\Delta^j + \frac{1}{2} a_{j,0} + \sum_{r=1}^{p} \left(a_{j,r} \cos\left(\frac{2r\pi t}{\Delta}\right) + b_{j,r} \sin\left(\frac{2r\pi t}{\Delta}\right) \right). \tag{2.28}$$

2.2 Representation of Stratonovich Stochastic Integrals

We write $\gamma = \frac{\pi}{\Delta}$, from definition,

$$J_{(0),t} = t, \quad J_{(0,0),t} = \frac{1}{2} t^2 \tag{2.29}$$

$$J_{(j),t} = \frac{1}{\Delta} W_\Delta^j J_{(0),t} + \frac{1}{2} a_{j,0} + \sum_{r=1}^{\infty} \left(a_{j,r} \cos(2\gamma r t) + b_{j,r} \sin(2\gamma r t) \right). \tag{2.30}$$

Then by integrating (2.28),

$$J_{(j,0),t} = \int_0^t J_{(j),s} ds$$
$$= \frac{1}{\Delta} W_\Delta^j J_{(0,0),t} + \frac{1}{2} a_{j,0} J_{(0),t} + \frac{\Delta}{2\pi} \sum_{r=1}^{\infty} \frac{1}{r} \left(a_{j,r} \sin(2\gamma rt) - b_{j,r}[\cos(2\gamma rt) - 1]\right). \tag{2.31}$$

So, by setting $t = \Delta$,

$$J_{(0)} = \Delta, \quad J_{(j)} = W_\Delta^j, J_{(0,0)} = \frac{1}{2}\Delta^2, \tag{2.32}$$
$$J_{(j,0)} = \frac{1}{2}\Delta \left(W_\Delta^j + a_{j,0}\right). \tag{2.33}$$

In addition, by (1.22),

$$J_{(0,j),\Delta} = J_{(j),t} J_{(0),\Delta} - J_{(j,0),\Delta} = \frac{1}{2}\Delta \left(W_\Delta^j - a_{j,0}\right). \tag{2.34}$$

Furthermore,

$$J_{(j_1,j_2)} = \int_0^\Delta J_{(j_1),s} \circ dW_s^{j_2} \tag{2.35}$$
$$= \frac{1}{\Delta} W_\Delta^{j_1} J_{(0,j_2)} + \frac{1}{2} a_{j_1,0} J_{(j_2)} + \int_0^\Delta \sum_{r=1}^{\infty} \left(a_{j_1,r} \cos(2\gamma rs) + b_{j_1,r} \sin(2\gamma rs)\right) \circ dW_s^{j_2} \tag{2.36}$$
$$= \frac{1}{2} W_\Delta^{j_1} W_\Delta^{j_2} - \frac{1}{2}\left(a_{j_2,0} W_\Delta^{j_1} - a_{j_1,0} W_\Delta^{j_2}\right) + \Delta A_{j_1,j_2}, \tag{2.37}$$

where,

$$A_{j_1,j_2} = \frac{1}{\Delta} \int_0^\Delta \sum_{r=1}^{\infty} \left(a_{j_1,r} \cos(2\gamma rs) + b_{j_1,r} \sin(2\gamma rs)\right) \circ dW_s^{j_2} \tag{2.38}$$
$$= \frac{1}{\Delta} \int_0^\Delta (\sum_{r=1}^{\infty} \left(a_{j_1,r} \cos(2\gamma rs) + b_{j_1,r} \sin(2\gamma rs)\right)) \tag{2.39}$$
$$\cdot (\frac{1}{\Delta} W_\Delta^{j_2} + 2\gamma \sum_{r=1}^{p} r \left(-a_{j_2,r} \sin(2\gamma rs) + b_{j_2,r} \cos(2\gamma rs)\right)) ds \tag{2.40}$$
$$= \frac{\pi}{\Delta} \sum_{r=1}^{\infty} r \left(a_{j_1,r} b_{j_2,r} - b_{j_1,r} a_{j_2,r}\right). \tag{2.41}$$

Based on this idea, we can derive,

$$J_{(0,0,0)} = \frac{1}{3!}\Delta^3, \quad J_{(0,j,0)} = \frac{1}{3!}\Delta^2 W_\Delta^j - \frac{1}{\pi}\Delta^2 b_j \tag{2.42}$$

$$J_{(j,0,0)} = \frac{1}{3!}\Delta^2 W_\Delta^j + \frac{1}{4}\Delta^2 a_{j,0} + \frac{1}{2\pi}\Delta^2 b_j \tag{2.43}$$

$$J_{(0,0,j)} = \frac{1}{3!}\Delta^2 W_\Delta^j - \frac{1}{4}\Delta^2 a_{j,0} + \frac{1}{2\pi}\Delta^2 b_j \tag{2.44}$$

with

$$b_j = \sum_{r=1}^{\infty} \frac{1}{r} b_{j,r} \tag{2.45}$$

$$J_{(j_1,0,j_2)} = \frac{1}{3!}\Delta W_\Delta^{j_1} W_\Delta^{j_2} + \frac{1}{2} a_{j_1,0} J_{(0,j_2)} + \frac{1}{2\pi}\Delta W_\Delta^{j_2} b_{j_1} - \Delta^2 B_{j_1,j_2} - \frac{1}{4}\Delta a_{j_2,0} W_\Delta^{j_1} + \frac{1}{2\pi}\Delta W_\Delta^{j_1} b_{j_2} \tag{2.46}$$

$$J_{(0,j_1,j_2)} = \frac{1}{3!}\Delta W_\Delta^{j_1} W_\Delta^{j_2} - \frac{1}{\pi}\Delta W_\Delta^{j_2} b_{j_1} + \Delta^2 B_{j_1,j_2} - \frac{1}{4}\Delta a_{j_2,0} W_\Delta^{j_1} + \frac{1}{2\pi}\Delta W_\Delta^{j_1} b_{j_2} + \Delta^2 C_{j_1,j_2} + \frac{1}{2}\Delta^2 A_{j_1,j_2} \tag{2.47}$$

with

$$B_{j_1,j_2} = \frac{1}{2\Delta} \sum_{r=1}^{\infty} \left(a_{j_1,r} a_{j_2,r} + b_{j_1,r} b_{j_2,r} \right) \tag{2.48}$$

and

$$C_{j_1,j_2} = -\frac{1}{\Delta} \sum_{\substack{r,l=1 \\ r \neq l}}^{\infty} \frac{r}{r^2 - l^2} \left(r a_{j_1,r} a_{j_2,l} + l b_{j_1,r} b_{j_2,l} \right) \tag{2.49}$$

$$J_{(j_1,j_2,0)} = J_{(j_1,j_2)} - J_{(j_1,0,j_2)} - J_{(0,j_1,j_2)} \tag{2.50}$$

2.3 Approximation

We using independent standard Gaussian random variables to represent,

$$\xi_j = \frac{1}{\sqrt{\Delta}} W_\Delta^j, \quad \zeta_{j,r} = \sqrt{\frac{2}{\Delta}} \pi r a_{j,r}, \quad \eta_{j,r} = \sqrt{\frac{2}{\Delta}} \pi r b_{j,r} \tag{2.51}$$

$$\mu_{j,p} = \frac{1}{\sqrt{\Delta \rho_p}} \sum_{r=p+1}^{\infty} a_{j,r}, \quad \phi_{j,p} = \frac{1}{\sqrt{\Delta \alpha_p}} \sum_{r=p+1}^{\infty} \frac{1}{r} b_{j,r} \tag{2.52}$$

where

$$\rho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^2} \quad \alpha_p = \frac{\pi^2}{180} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^4}. \tag{2.53}$$

Here p decides the truncated order and is set to be large enough.

For example,

$$J^p_{(0)} = \Delta, \quad J^p_{(j)} = \sqrt{\Delta}\xi_j, \quad J^p_{(0,0)} = \frac{1}{2}\Delta^2 \tag{2.54}$$

$$J^p_{(j,0)} = \frac{1}{2}\Delta\left(\sqrt{\Delta}\xi_j + a_{j,0}\right), \quad J^p_{(0,j)} = \frac{1}{2}\Delta\left(\sqrt{\Delta}\xi_j - a_{j,0}\right) \tag{2.55}$$

where

$$a_{j,0} = -\frac{1}{\pi}\sqrt{2\Delta}\sum_{r=1}^{p}\frac{1}{r}\zeta_{j,r} - 2\sqrt{\Delta\rho_p}\mu_{j,p} \tag{2.56}$$

$$J^p_{(j_1,j_2)} = \frac{1}{2}\Delta\xi_{j_1}\xi_{j_2} - \frac{1}{2}\sqrt{\Delta}\left(a_{j_2,0}\xi_{j_1} - a_{j_1,0}\xi_{j_2}\right) + \Delta A^p_{j_1,j_2}, \tag{2.57}$$

with

$$A^p_{j_1,j_2} = \frac{1}{2\pi}\sum_{r=1}^{p}\frac{1}{r}\left(\zeta_{j_1,r}\eta_{j_2,r} - \eta_{j_1,r}\zeta_{j_2,r}\right). \tag{2.58}$$

3 Project IV: Due May 12 before lecture

IV-1: Given 1D process driven by m-dimension Wiener process,

$$dX = adt + \sum_{j=1}^{m} b^j dW^j, \tag{3.59}$$

determine $\underline{f}_{(j_1,j_2,j_3,j_4)}$ and $f_{(j_1,j_2,j_3,j_4)}$ for $j_1, \ldots, j_4 \in \{1, \ldots, m\}$.

IV-2: Approximating $W_1^1 I_{(1,2)}[1]_{0,1}$

1. Find a representation of $I_{(1,2)}$ in terms of some random coefficients.
2. Calculate $E(W_1^1 I_{(1,2)}[1]_{0,1})$ based on a truncation in (1) with Monte-Carlo.
3. Calculate $E(W_1^1 I_{(1,2)}[1]_{0,1})$ by direct sampling paths of Wiener process and integrating along path.