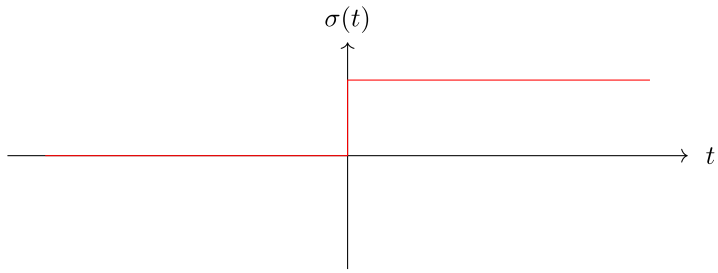
Math 18500 Week 9: Impulse Response and Transfer Functions

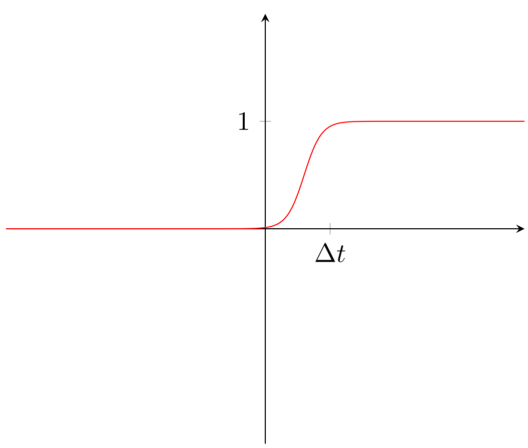
Transforms of Piecewise Functions. Consider the function

$$\sigma(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

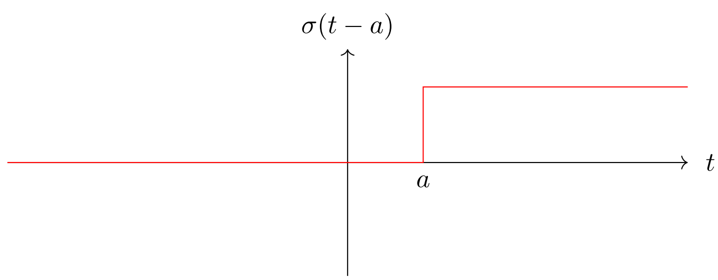
This function (called the *unit step function*) can be used to model something like the voltage supplied to a circuit when we "flip a switch" at $t = 0$. Its graph looks like this:



Of course, this function cannot be a realistic model for any actual physical phenomenon - physical quantities do not change discontinuously. In any realistic situation there would be a very short time period Δt (on the order of microseconds or nanoseconds, perhaps) during which the transition from 0 to 1 occurred continuously. So, it is a good idea to think of $\sigma(t)$ as a function whose graph looks like this, where Δt is extremely small:



To model a function which turns on rapidly at $t = a$, we can use the function $\sigma(t-a)$:



The Laplace transform of $\sigma(t-a)$ is given by

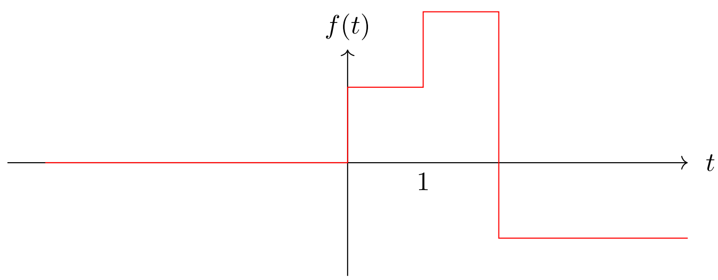
$$\mathcal{L}\{\sigma(t-a)\} = \int_0^\infty \sigma(t-a)e^{-st} = \int_a^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-sa}}{s}$$

Notice that if $a = 0$, we get

$$\mathcal{L}\{\sigma(t)\} = \frac{1}{s} = \mathcal{L}\{1\}.$$

This is because the Laplace transform of a function $f(t)$ only takes into account the values of $f(t)$ for $t > 0$. If two functions agree on the interval $[0, \infty)$, like $\sigma(t)$ and 1 do, then the Laplace transform does not distinguish them. This does not contradict the Laplace inversion theorem, which only says that the values of $f(t)$ *where* t *is greater than* 0 can be recovered from $F(s)$.

Now suppose we want to compute the Laplace transform of a function $f(t)$ whose graph looks like this:



This can be done by *writing* $f(t)$ *as a linear combination of step functions*:

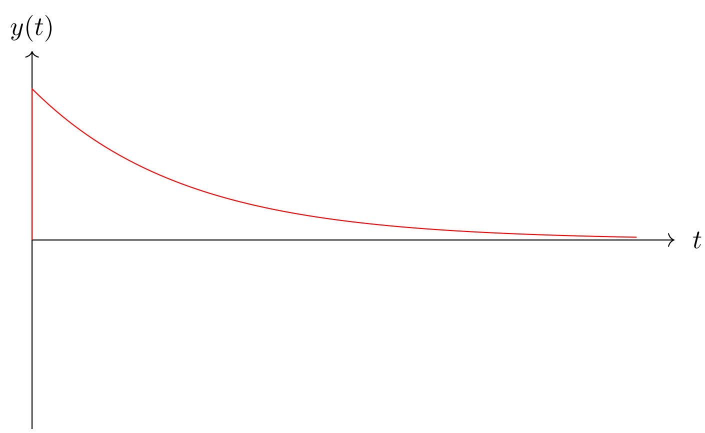
$$f(t) = \sigma(t) + \sigma(t-1) - 3\sigma(t-2)$$

To understand what makes this formula work, think about what each term does. For $t < 0$, all three terms are zero. At $t = 0$ the first term "turns on" and the value of the function jumps up to 1. At $t = 1$, the second term turns on and the value of the function jumps up to $1 + 1 + 0 = 2$. At $t = 2$, the third terms turns on and the value jumps down to $1 + 1 - 3 = -1$.

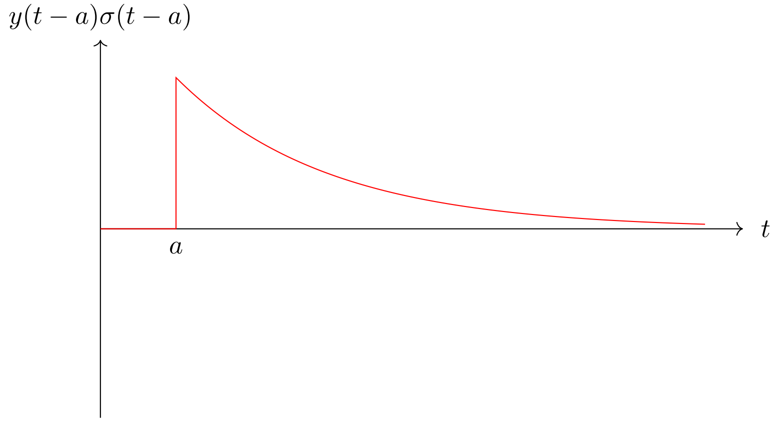
Once we have such a formula, we can get the Laplace transform easily by applying the linearity property of the Laplace transform:

$$\begin{aligned} \mathcal{L}\{\sigma(t) + \sigma(t-1) - 3\sigma(t-2)\} &= \mathcal{L}\{\sigma(t)\} + \mathcal{L}\{\sigma(t-1)\} - 3\mathcal{L}\{\sigma(t-2)\} \\ &= \frac{1}{s} + \frac{e^{-s}}{s} - 3\frac{e^{-2s}}{s} \\ &= \frac{1 + e^{-s} - 3e^{-2s}}{s} \end{aligned}$$

The formula for the Laplace transform of $\sigma(t-a)$ can be generalized as follows. Consider a function of the form $y(t-a)\sigma(t-a)$, where $y(t)$ is defined on $[0, \infty)$ and $a > 0$. To think visualize a function of this form, we imagine "suddenly turning on $y(t)$ at time $t = a$, with a delay of a units of time". So, if the graph of $y(t)$ looks like this:



then the graph of $y(t-a)\sigma(t-a)$ would look like this:



The Laplace transform of $y(t-a)\sigma(t-a)$ is given by

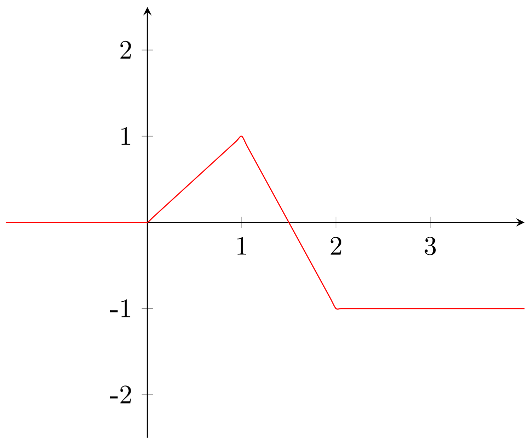
$$\mathcal{L}\left\{y(t-a)\sigma(t-a)\right\} = \int_0^\infty y(t-a)\sigma(t-a)e^{-st}dt = \int_a^\infty y(t-a)e^{-st}dt.$$

Making the substitution $\tau = t-a$, $d\tau = dt$, we see that

$$\mathcal{L}\left\{y(t-a)\sigma(t-a)\right\} = \int_0^\infty y(\tau)e^{-s(\tau+a)}d\tau = e^{-sa}\int_0^\infty y(\tau)e^{-s\tau}d\tau = e^{-sa}\mathcal{L}\left\{y(t)\right\}$$

So, when we introduce a delay of a units of time, the Laplace transform gets multiplied by e^{-sa}.

This result can be used to calculate the Laplace transform of any function with a piecewise definition. For example, consider the function $g(t)$ whose graph looks like this:



This function is given by the following piecewise formula:

$$g(t) = \begin{cases} t & 0 < t < 1 \\ 3-2t & 1 < t < 2 \\ -1 & t > 2 \end{cases}$$

This function can be written in terms of step functions, as follows:

$$g(t) = t\sigma(t) + ((3-2t)-t)\sigma(t-1) + (-1-(3-2t))\sigma(t-2).$$

$$g(t) = t\sigma(t) - 3(t-1)\sigma(t-1) + 2(t-2)\sigma(t-2)$$

Therefore, its Laplace transform is

$$G(s) = \frac{1}{s^2} - \frac{3e^{-s}}{s^2} + \frac{2e^{-2s}}{s^2}$$

Impulse Response and the Transfer Function. Often we want to know how an oscillator will respond to an input which is delivered over a very short period of time. An input like this is called an *impulse*, and the response of the oscillator is called the *impulse response* of the oscillator.

For, example consider about an electronic circuit with a resistor, a capacitor, an inductor, a 1-volt battery, and a switch. Recall that the current through a circuit like this can be modeled by a second order equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = V'(t).$$

where the forcing term is the *derivative* of the voltage. If we flip the switch at time $t = 0$ and "turn on" the circuit, then we might want to model $V(t)$ using a step function:

$$V(t) = \sigma(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

But this presents us with a problem - what function should we use to model the *derivative* of the voltage?

Naively, we might compute the derivative of $\sigma(t)$ like this:

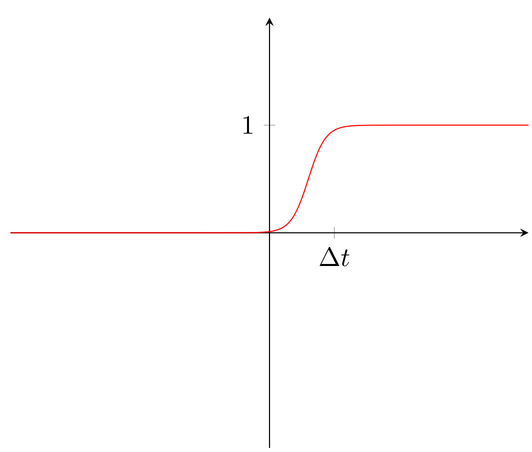
$$\sigma'(t) = \begin{cases} 0 & t < 0 \\ 0 & t > 1 \end{cases} = 0$$

But this would violate the fundamental theorem of calculus, which says that

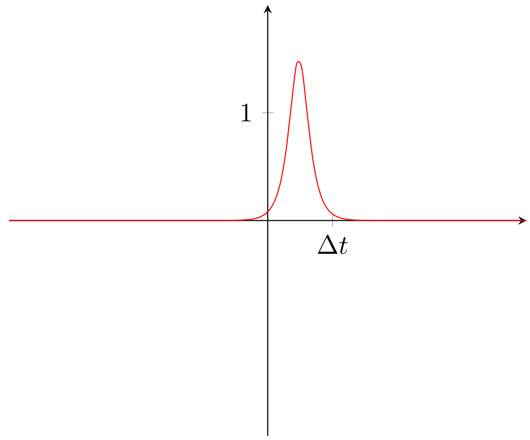
$$\int_{-1}^{1} \sigma'(t)dt = \sigma(t)\,\Big|_{-1}^{1} = 1 - 0 = 1.$$

At this point a mathematician might say, well, $\sigma(t)$ isn't differentiable at $t = 0$, so technically the fundamental theorem of calculus doesn't apply to it.

But the real problem is that $\sigma(t)$ isn't an appropriate function to use in our model - as we have discussed, $\sigma(t)$ is really an idealized version of a function which actually looks like this:



This function *is* differentiable, and its derivative looks like this:



The graph of the derivative has a spike during the period of transition, and by the fundamental theorem of calculus, the area under this spike must be exactly equal to 1.

Thinking in this way leads to a way out of the problem: we can formally introduce a "function" which satisfies both of the following properties:

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_a^b \delta(u)du = \begin{cases} 1 & a \leq 0 \leq b \\ 0 & \text{else} \end{cases}$$

No *actual* function can satisfy both properties, so $\delta(t)$ cannot *really* be a function - mathematicians refer to mathematical constructions like this as *generalized functions* or *distributions*.

For the most part, it is only valid to use generalized functions when they appear *inside of integrals*. For example, consider an integral of the form

$$\int_{t_0}^{t_1} f(t)\delta(t-a)dt$$

where $t_0 \leq a \leq t_1$. To evaluate an integral like this, we might try to integrate by parts:

$$\begin{aligned}
\int_{t_0}^{t_1} f(t)\delta(t-a)dt &= f(t)\sigma(t-a)|_{t_0}^{t_1} - \int_{t_0}^{t_1} f'(t)\sigma(t-a) \\
&= (f(t_1) - 0) - \int_a^{t_1} f'(t)dt \\
&= f(t_1) - (f(t_1) - f(a)) \\
&= f(a)
\end{aligned}$$

This identity,

$$\int_{t_0}^{t_1} f(t)\delta(t-a)dt = f(a),$$

is valid for *any* function $f(t)$ and any value of a with $t_0 \leq a \leq t_1$. Some people take it to be the *defining feature* of the delta function.

We can use the identity above to compute the "Laplace transform of a delta function":

$$\mathcal{L}\left\{\delta(t-a)\right\} = \int_0^\infty \delta(t-a)e^{-st}dt = e^{-sa}$$

Fortunately, this is the only fact that we need to solve differential equations involving delta functions. Indeed, suppose that we want to solve an initial value problem

$$my'' + by' + ky = \delta(t-a), \quad y(0) = y'(0) = 0,$$

i.e. we want to model the response of an oscillator to a *unit impulse* at time $t = a$.

Taking the Laplace transform of both sides, we see that

$$(ms^2 + bs + k)Y(s) = e^{-sa},$$

so

$$Y(s) = \frac{e^{-sa}}{ms^2 + bs + k}.$$

Therefore, the solution is

$$y(t) = h(t-a)\sigma(t-a),$$

where

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{ms^2 + bs + k}\right\}.$$

This function $h(t)$ is called the *impulse response function* of the oscillator. Its Laplace transform,

$$H(s) = \frac{1}{ms^2 + bs + k}$$

is called the *transfer function* of the oscillator.

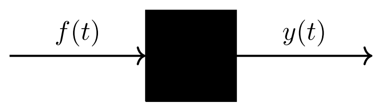
To understand the significance of the transfer function, consider a general initial value problem with *initial rest conditions*

$$my'' + by' + ky = f(t), \quad y(0) = y'(0) = 0.$$

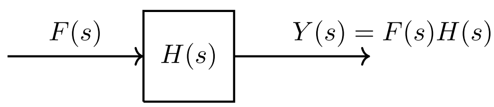
The Laplace transform of the solution can be found easily:

$$Y(s) = \frac{F(s)}{ms^2 + bs + k} = F(s)H(s).$$

Therefore, the "black box" model



becomes completely transparent *in the frequency domain*:



To "transfer" an input through the box, we *multiply by the transfer function.*

One application of the transfer function is that it encodes the *frequency response* of the system. To understand this, suppose we take the input $f(t)$ to be a sinusoid,

$$f(t) = e^{i\omega t}$$

Then the Laplace transform of the response will be

$$Y(s) = H(s)F(s) = \frac{H(s)}{s - i\omega}$$

To find the solution, we can apply a partial fractions decomposition:

$$\frac{H(s)}{s - i\omega} = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2} + \frac{C}{s - i\omega}$$

and take the inverse Laplace transform:

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + Ce^{i\omega t}$$

Here λ_1 and λ_2 are roots of the auxiliary equation,

$$m\lambda^2 + l\lambda + k = 0,$$

so the terms involving A and B are transient. The value of C can be found using the cover-up method. Multiplying by $s - i\omega$ gives

$$H(s) = \frac{A(s - i\omega)}{s - \lambda_1} + \frac{B(s - i\omega)}{s - \lambda_2} + C$$

and from this we can obtain C, by setting $s = i\omega$:

$$C = H(i\omega)$$

Therefore the steady state response is

$$Y(s) = H(i\omega)e^{i\omega t},$$

and the *amplitude* of the steady state response is

$$|H(i\omega)|.$$

This is called the *frequency response function* of the oscillator.

The frequency response function is something we are already familiar with - we encountered it in the form

$$\frac{1}{|m(i\omega)^2 + l(i\omega) + k|} = \frac{1}{\sqrt{(k - m\omega^2)^2 + l^2\omega^2}}.$$

Thinking in terms of the transfer function gives us a more enlightening way of remembering this formula!

Convolutions. At this point, we still do not know how to solve initial value problems

$$my'' + by' + ky = f(t), \; , \quad y(0) = y_0 \; , \quad y'(0) = v_0$$

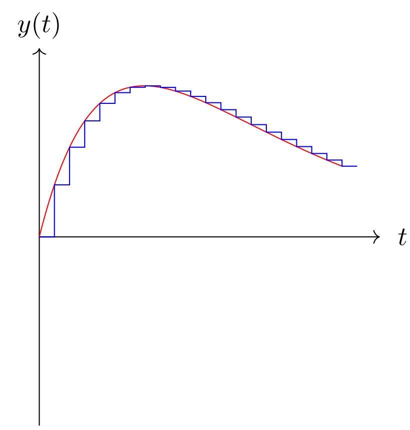
where the right hand side is an *arbitrary* function of t. We know how to do this in a number of special cases by guessing, or using Laplace transforms, but we don't have a *general formula.*

Such a formula can be derived using the concept of a delta function. For any function $f(t)$ which is defined on the interval $[0, \infty)$, we have:

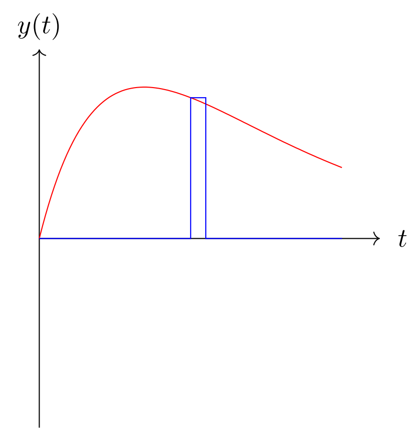
$$f(t) = \int_0^\infty f(u)\delta(t-u)du$$

The integral on the right hand side can be thought of as a sum of functions of t, indexed by a parameter u. Intuititively, it tells us that *every function is a superposition of delta functions.*

There is also a geometric way to visualize the idea that every function is a sum of delta functions. Notice that we can approximate any continuous function (red) using a *piecewise constant* function (blue):



In turn, any piecewise constant function can be viewed as a sum of functions which are zero except on a small interval of width Δt, like this one:



In the limit as $\Delta t \to 0$, the function shown above can be approximated by $\delta(t-u)du$, and the sum can be replaced by an integral - hence,

$$f(t) = \int_0^\infty f(u)\delta(t-u)du$$

Now recall that the solution of the equation

$$my_u'' + by_u' + ky_u = \delta(t-u), \quad y(0) = 0, \; y'(0) = 0$$

is given by the formula

$$y_u(t) = h(t-u)\sigma(t-u),$$

where $h(t)$ is the impulse response function. Therefore, to solve the equation

$$my'' + by' + ky = f(t) = \int_0^\infty f(u)\delta(t-u)du, \;\; y(0) = y'(0) = 0,$$

we can apply the superposition principle (which is just as valid for integrals as it is for sums):

$$y(t) = \int_0^\infty f(u)y_u(t)du = \int_0^\infty f(u)h(t-u)\sigma(t-u)du$$

The integrand above is only nonzero for $u < t$, so:

$$y(t) = \int_0^t f(u)h(t-u)du.$$

This is a closed formula for the solution of an arbitrary second order equation with constant coefficients.

Now, maybe you are skeptical of this application of the superposition principle, or maybe you found the argument confusing - we therefore give a second derivation. In general, given two functions $f(t)$ and $g(t)$ we can form their *convolution*:

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

The convolution of $f(t)$ and $g(t)$ is a new function of t which incorporates the values of both functions. Convolution is therefore an *operation on functions*, just like integration or differentiation, but instead of taking a single function as its input, it takes two functions and "mashes them together" into a new function.

You would be forgiven for finding this operation to be somewhat convoluted (hence the name!). However, the significance of convolution lies in the fact that *the Laplace transform of a convolution is a product*:

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s).$$

Alternatively, the *inverse Laplace transform of a product is a convolution*:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t).$$

This important fact can be derived by reversing the order of integration in a double integral:

$$\begin{aligned}
\mathcal{L}\{f(t) * g(t)\} &= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st}dudt \\
&= \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st}dtdu \\
&= \int_0^\infty f(u)e^{-su} \int_u^\infty g(t-u)e^{-s(t-u)}dtdu \\
&= \int_0^\infty f(u)e^{-su} \int_0^\infty g(v)e^{-sv}dvdu \\
&= \int_0^\infty f(u)e^{-su}G(s)du \\
&= F(s)G(s)
\end{aligned}$$

Now suppose we want to solve an initial value problem with initial rest conditions:

$$my'' + by' + ky = f(t), \;\; y(0) = y'(0) = 0.$$

We can do this by taking the Laplace transform of both sides:

$$(ms^2 + bs + k)Y(s) = F(s)$$

$$Y(s) = \frac{F(s)}{ms^2 + bs + k} = F(s)H(s).$$

Taking the inverse Laplace transform, we obtain the same result as before:

$$y(t) = \mathcal{L}^{-1}\{F(s)H(s)\} = f(t) * h(t) = \int_0^t f(u)h(t-u)du.$$

Green's Formula. In this optional section, we will explain how to solve arbitrary linear equations

$$\mathcal{O}[y(t)] = f(t),$$

where $\mathcal{O}$ is a second order differential operator

$$\mathcal{O} = m(t)\frac{d^2}{dt^2} + l(t)\frac{d}{dt} + k(t).$$

with *nonconstant* coefficients. Equations of this form might be used to model a mass-spring system whose properties change over time. For example, the mass on the spring might be a sandbag which is leaking sand, or the spring constant might decrease over time due to the spring being worn out.

Unfortunately, there is no general method for solving *homogeneous* second order equations, without resorting to power series or numerical solutions. However, there is a general method for solving *inhomogeneous* equations, provided that we already know all of the solutions of the corresponding homogeneous equation.

This general method is called *variation of parameters*. To explain how it works, assume that we are given two functions y_1 and y_2, which are a fundamental pair of solutions of the homogeneous equation

$$m(t)y'' + l(t)y' + k(t)y = 0, \tag{1}$$

in the sense that an arbitrary solution of the homogeneous equation takes the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some values of c_1 and c_2.

Then to find a particular solution of the inhomogeneous equation

$$m(t)y'' + l(t)y' + k(t)y = f(t), \tag{2}$$

we make an ansatz of the form

$$y = v_1 y_1 + v_2 y_2, \tag{3}$$

where $v_1 = v_1(t)$ and $v_2 = v_2(t)$ are unknown functions (the *varying parameters*).

This appears to make the problem we are trying to solve even more difficult, since now we must solve for *two* unknown functions v_1 and v_2 instead of the *single* unknown function y! However, we will immediately remove this extra degree of freedom by imposing a simplifying constraint on v_1 and v_2.

To substitute the ansatz (3) into equation (2), we first compute the derivatives of y using the product rule:

$$\begin{aligned} y &= (v_1 y_1 + v_2 y_2) \\ y' &= (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2) \\ y'' &= (v_1 y_1'' + v_2 y_2'') + (v_1' y_1' + v_2' y_2') + (v_1' y_1 + v_2' y_2)' \end{aligned}$$

To simplify this, we can impose the following constraint on v_1 and v_2,

$$v_1' y_1 + v_2' y_2 = 0 \tag{4}$$

With this simplifying constraint, the derivatives of y become

$$\begin{aligned} y &= (v_1 y_1 + v_2 y_2) \\ y' &= (v_1 y_1' + v_2 y_2') \\ y'' &= (v_1 y_1'' + v_2 y_2'') + (v_1' y_1' + v_2' y_2') \end{aligned}$$

If we then substitute into (2), we get

$$m(v_1' y_1' + v_2' y_2') + m(v_1 y_1'' + v_2 y_2'') + l(v_1 y_1' + v_2 y_2') + k(v_1 y_1 + v_2 y_2) = f.$$

Since y_1 and y_2 are solutions of (1), the last three terms on the left hand side cancel, leaving the equation

$$v_1' y_1' + v_2' y_2' = \frac{f}{m}.$$

We therefore have a system of two equations for v_1' and v_2':

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= \frac{f}{m} \end{aligned}$$

To solve this system of equations, we multiply the first equation by y_2', multiply the second equation by y_2, and subtract:

$$v_1'(y_1y_2' - y_1'y_2) = -\frac{f}{m}$$

Solving for v_1', we find that

$$v_1' = -\frac{f}{m(y_1y_2' - y_1'y_2)} = -\frac{y_2 f}{mW}$$

where $W = y_1y_2' - y_2y_1'$ is an auxilliary function called the *Wronskian* of y_1 and y_2.

Integrating, we get a particular solution $v_1(t)$ which satisfies $v_1(t_0) = 0$:

$$v_1 = -\int_{t_0}^{t} \frac{y_2(u)f(u)}{m(u)W(u)}du.$$

Similarly, we can find a solution v_2 which satisfies $v_2(0) = 0$:

$$v_2 = \int_{t_0}^{t} \frac{y_1(u)f(u)}{m(u)W(u)}du$$

Therefore, one particular solution of the inhomogeneous equation is

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = \int_{t_0}^{t} \frac{y_1(u)y_2(t) - y_1(t)y_2(u)}{m(u)\left(y_1(u)y_2'(u) - y_2(u)y_1'(u)\right)} f(u)du \tag{5}$$

This solution satisfies the simplest possible initial conditions

$$\begin{aligned} y_p(t_0) &= v_1(t_0)y_1(t_0) + v_2(t_0)y_2(t_0) = 0 \\ y_p'(t_0) &= v_1'(t_0)y_1(t_0) + v_1(t_0)y_1'(t_0) + v_2'(t_0)y_2(t_0) + v_2(t_0)y_2(t_0) \\ &= v_1'(t_0)y_1(t_0) + v_2'(t_0)y_2(t_0) = 0 \end{aligned}$$

where in the last line above we have used the assumption (4).

The function which appears inside the integral in (5),

$$G(t,u) = \frac{y_1(u)y_2(t) - y_1(t)y_2(u)}{m(u)\left(y_1(u)y_2'(u) - y_2(u)y_1'(u)\right)},$$

is called *Green's function* for the operator $\mathcal{O}$. We will refer to the formula

$$y_p(t) = \int_{t_0}^{t} G(t,u)f(u)du.$$

as *Green's formula* (but this terminology is not standard). It is a generalization of the formula

$$y_p(t) = \int_{t_0}^{t} h(t-u)f(u)du$$

which we derived previously for operators with *constant* coefficients.

A technical caveat is now in order. Green's formula is only valid if we make two assumptions:

(1) The Wronskian of y_1 and y_2 must be nonzero.
(2) The leading coefficient $m(t)$ in the operator $\mathcal{L}$ must be nonzero.

These conditions together guarantee that there is no division by zero in equation (5).

A linear differential operator $\mathcal{O}$ which satisfies the condition (2) is said to be *regular*. In general, solutions of linear differential equations are only guaranteed on intervals where the corresponding operator is regular.

Notice that in Green's formula we only need values of $G(t,u)$ for $u \leq t$. In fact, it makes sense to define

$$G(t,u) = \begin{cases} 0 & t < u \\ \frac{y_1(u)y_2(t) - y_1(t)y_2(u)}{m(u)\left(y_1(u)y_2'(u) - y_2(u)y_1'(u)\right)} & t \geq u \end{cases}$$

With this definition, Green's formula can also be written as

$$y_p(t) = \int_{t_0}^{\infty} G(t,u)f(u)du$$

To better understand why it is a good idea to make this definition, say we want to solve the equation

$$m(t)y'' + l(t)y' + k(t)y = \delta(t - t_0)$$

i.e. we want to obtain the response to an impulse at time $t = t_0$. Then we can apply Green's formula:

$$y(t) = \int_0^t G(t,u)\delta(u - t_0)du = G(t, t_0)$$

This result explains the physical meaning of the Green's function - for any fixed value of u, the Green's function is the response of the oscillator to an impulse at time $t = u$. It also explains why it's a good idea to define $G(t, u) = 0$ for $t < u$ - the oscillator should be at rest before the impulse occurs.

Finally, another way to understand the role of the Green's function is to introduce the *integral operator*

$$\mathcal{I}[f(t)] = \int_{t_0}^t G(t,u)f(u)du.$$

With this notation, what we have showed above is that

$$\mathcal{O}[\mathcal{I}[f(t)]] = f(t),$$

for any function $f(t)$. This is analogous to the fundamental theorem of calculus, which states that

$$\frac{d}{dt}\int_{t_0}^t f(u)du = f(t).$$

From this point of view, Green's formula is a vast generalization of the fundamental theorem of calculus!