Lecture 14: Implicit Strong Schemes

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Abstract

Introducing implicit schemes and discuss on stability.

1 Implicit Schemes

1.1 Implicit Strong Taylor Schemes

Implicit Euler:

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta + b\Delta W, \tag{1.1}$$

$\Delta W = W_{t_{n+1}} - W_{t_n}$, $b = b(t_n, Y_n)$.

Note that the noisy part is explicit still, the scheme is strictly speaking semi-implicit. Fully implicit scheme will run into the inverse of a Gaussian r.v, infinite moments etc.

> Recall fully implicit Euler for multiplicative noise case:
>
> $$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta + b(t_{n+1}, Y_{n+1})\Delta W, \tag{1.2}$$
>
> applied to
>
> $$dX_t = \lambda X_t dt + X_t dW_t,$$
>
> yields:
>
> $$Y_n = Y_0 \prod_{k=0}^{n-1} \frac{1}{1 - \lambda\Delta - \Delta W}, \tag{1.3}$$
>
> is not suitable for strong approximation as the denominator can be zero.

More generaly,

$$Y_{n+1} = Y_n + [\alpha a(t_{n+1}, Y_{n+1}) + (1-\alpha)a]\Delta + b\Delta W, \tag{1.4}$$

$\alpha \in [0, 1]$. The $\alpha = 1/2$ extends trapezoidal method in deterministic case. Implicit Euler is order 0.5 accurate.

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Implicit Milstein scheme:

$$\begin{aligned} Y_{n+1} &= Y_n + a(t_{n+1}, Y_{n+1})\Delta + b\Delta W \\ &\quad + \frac{1}{2} bb'[(\Delta W)^2 - \Delta], \end{aligned} \tag{1.5}$$

order 1 accurate.

Implicit Order 1.5 scheme:

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}[a(Y_{n+1}) + a]\Delta + b\Delta W \\ &\quad + a'b(\Delta Z - \frac{1}{2}\Delta W \Delta) \\ &\quad + (ab' + \frac{1}{2} b^2 b'')(\Delta W \Delta - \Delta Z) \\ &\quad + \frac{1}{2} bb'[(\Delta W)^2 - \Delta] \\ &\quad + \frac{1}{2} b(bb')'[\frac{1}{3}(\Delta W)^2 - \Delta]\Delta W, \end{aligned} \tag{1.6}$$

the last three terms same as those in order 1.5 Ito-Taylor scheme. $\Delta Z = I_{(1,0)}$. Applying the Ito formula:

$$\begin{aligned} f(X_{t+\Delta}) &= f(X_t) + [a(X_t) f'(X_t) + \frac{1}{2} b^2(X_t) f''(X_t)]\Delta \\ &\quad + b(X_t) f'(X_t)\Delta W, \end{aligned} \tag{1.7}$$

to $a(Y_{n+1})$,

$$a(Y_{n+1}) = a + (aa' + \frac{1}{2} a'' b^2)\Delta + a' b \Delta W \tag{1.8}$$

(1.6) is same as order 1.5 Ito-Taylor scheme:

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta W + a'b\Delta Z \\ &\quad + \frac{1}{2}(aa' + b^2 a''/2)\Delta^2 \\ &\quad + (ab' + \frac{1}{2} b^2 b'')(\Delta W \Delta - \Delta Z) \\ &\quad + \frac{1}{2} bb'[(\Delta W)^2 - \Delta] \\ &\quad + \frac{1}{2} b(bb')'[\frac{1}{3}(\Delta W)^2 - \Delta]\Delta W, \end{aligned} \tag{1.9}$$

where $\cdots$ = last three terms in (1.6).

1.2 Implicit RK

1st Order Recall Platen's scheme:

$$Y_{n+1} = Y_n + a\Delta + b\Delta W_n + \frac{1}{2\sqrt{\Delta}} \left(b\left(Y_n^*\right) - b\right)\left(\left(\Delta W_n\right)^2 - \Delta\right) \tag{1.10}$$

Then Implicit order 1 strong RK:

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta + b\Delta W + \frac{1}{2\sqrt{\Delta}}(b(t_n, Y_n^*) - b)[(\Delta W)^2 - \Delta)], \tag{1.11}$$

$Y_n^* = Y_n + a\Delta + b\sqrt{\Delta}$.

1.5 Order Explicit Order 1.5 RK Scheme yields,

$$\begin{aligned} Y_{n+1} = {} & Y_n + \frac{1}{4}[a(Y_+) + 2a + a(Y_-)]\Delta + b\Delta W, \\ & + \frac{1}{2\sqrt{\Delta}}[a(Y_+) - a(Y_-)]\Delta Z \\ & + \frac{1}{4\sqrt{\Delta}}[b(Y_+) - b(Y_-)][(\Delta W)^2 - \Delta)] \\ & + \frac{1}{2\Delta}[b(Y_+) - 2b + b(Y_-)][\Delta W\Delta - \Delta Z) \\ & + \frac{1}{4\Delta}[b(G_+) - b(G_-) - b(Y_+) + b(Y_-)] \times [\frac{1}{3}(\Delta W)^2 - \Delta]\Delta W, \end{aligned} \tag{1.12}$$

with:

$$Y_{\pm} = Y_n + a\Delta \pm b\sqrt{\Delta},$$

and

$$G_{\pm} = Y_+ \pm b(Y_+)\sqrt{\Delta}.$$

Replacing derivatives by finite difference to get order 1.5 implicit RK:

$$\begin{aligned} Y_{n+1} = {} & Y_n + \frac{1}{2}[a(Y_{n+1}) + a]\Delta + b\Delta W, \\ & + \frac{1}{2\sqrt{\Delta}}[a(Y_+) - a(Y_-)](\Delta Z - \frac{1}{2}\Delta W\Delta) \\ & + \frac{1}{4\sqrt{\Delta}}[b(Y_+) - b(Y_-)][(\Delta W)^2 - \Delta)] \\ & + \frac{1}{2\Delta}[b(Y_+) - 2b + b(Y_-)](\Delta W\Delta - \Delta Z) \\ & + \frac{1}{4\Delta}[b(G_+) - b(G_-) - b(Y_+) + b(Y_-)] \times [\frac{1}{3}(\Delta W)^2 - \Delta]\Delta W, \end{aligned} \tag{1.13}$$

1.3 Implicit 2-step

good for solving stiff eqns, order 1 implicit two step strong scheme:

$$Y_{n+1} = Y_{n-1} + [a(t_{n+1}, Y_{n+1}) + a]\Delta + V_n + V_{n-1}, \tag{1.14}$$

with:

$$V_n = b\Delta W_n + \frac{1}{2}bb'[(\Delta W_n)^2 - \Delta].$$

2 Absolute Stability

Apply a scheme to SDE:

$$dX_t = \lambda X_t dt + dW_t, \tag{2.15}$$

$Re(\lambda) < 0$ to get:

$$Y_{n+1} = G(\lambda\Delta)Y_n + Z_n, \tag{2.16}$$

where Z_n are r.v independent of history and λ. Absolute stability is the region of $\lambda\Delta$ such that $|G(\lambda\Delta)| < 1$. A-stable means left half plane is absolutely stable region.

Fact: implicit Euler and Milstein, and order 1.5 RK are A-stable.

2.1 Example: Euler Scheme

For explicit Euler scheme:

$$Y_{n+1} = Y_n(1 + \lambda\delta) + W_{n+1} - W_n$$

absolute-stable if:

$$|G(\lambda\Delta)| = |1 + \lambda\delta| < 1, \mathrm{Re}(\lambda) < 0.$$

So the region is a open disc of unit radius centered on the point $(-1, 0)$.

For implicit Euler,

$$Y_{n+1} = Y_n + [\alpha a(t_{n+1}, Y_{n+1}) + (1-\alpha)a]\Delta + b\Delta W \tag{2.17}$$

So

$$G(\lambda\Delta) = \frac{1 + (1-\alpha)\lambda\Delta}{1 - \alpha\lambda\Delta}. \tag{2.18}$$

$|G(\lambda\Delta)| < 1$ is equivalent to,

$$|1 + (1-\alpha)\lambda\Delta|^2 < |1 - \alpha\lambda\Delta|^2 \tag{2.19}$$

that in turn is equivalent to

$$(1 - 2\alpha)\left(\lambda_1^2 + \lambda_2^2\right)\Delta^2 + 2\lambda_1\Delta < 0$$

where $\lambda = \lambda_1 + \imath\lambda_2$.

- For $\frac{1}{2} \leq \alpha \leq 1$ this is satisfied by all λ with $\lambda_1 < 0$, so the schemes are then A-stable.
- For $0 \leq \alpha < \frac{1}{2}$ we can write as

$$(\lambda_1 \Delta + A)^2 + (\lambda_2 \Delta)^2 < A^2$$

where $A = (1 - 2\alpha)^{-1}$, so the region of absolute stability is the interior of the circle with radius A and centered at $-A + 0\imath$.

3 Convergence Proofs

Recall the convergence theorem for strong Ito scheme. We need the scheme can be represent by,

$$Y_{n+1} = Y_n + \sum_{\alpha \in A_\gamma} g_{\alpha,n} I_\alpha + R_n. \tag{3.20}$$

Now consider Ito-Taylor expansion of X_t, $X_s = x$:

$$X_{s+\Delta} = x + a\Delta + \frac{1}{2} L^0 a \Delta^2 + M_\Delta(s), \tag{3.21}$$

$$L^0 = a\partial\alpha_x + \frac{1}{2} b^2 \partial\alpha^2_{xx},$$

M_Δ contains noisy terms.

The key part is we only expand terms fully deterministic to avoid inverting terms include randomness, note

$$a(s,x) = a(s+\Delta, X_{s+\Delta}) - L^0 a\Delta + N_\Delta(s), \tag{3.22}$$

$$N_\Delta(t) = -\frac{1}{2} L^0 L^0 a \Delta^2 - \left(L^1 a \Delta W + L^0 L^1 a I_{(0,1)} + L^1 L^0 a I_{(1,0)}\right) \tag{3.23}$$

$$- L^1 L^1 a I_{(1,1)} - L^1 L^1 L^1 a I_{(1,1,1)} + \tilde{R}_{1.5} \tag{3.24}$$

Likewise,

$$L^0 a(s,x) = L^0 a(s+\Delta, X_{s+\Delta}) + P_\Delta, \tag{3.25}$$

$$P_\Delta = -L^0 L^0 a \Delta - L^1 L^0 a \Delta W + \tilde{R}_1 \tag{3.26}$$

For any α, $\beta \in [0,1]$, plug (3.22) and (3.25) into (3.21):

$$\begin{aligned} X_{s+\Delta} &= x + \alpha a \Delta + (1-\alpha) a \Delta + \frac{1}{2} L^0 a \Delta^2 + M_\Delta \\ &= x + [\alpha(a(s+\Delta, X_{s+\Delta}) + N_\Delta(s)) + (1-\alpha)a]\Delta \\ &\quad +(1/2 - \alpha) L^0 a \Delta^2 + M_\Delta(s) \\ &= x + [\alpha(a(s+\Delta, X_{s+\Delta}) + N_\Delta(s)) + (1-\alpha)a]\Delta \\ &\quad +(1/2 - \alpha)[\beta L^0 a + (1-\beta) L^0 a]\Delta^2 + M_\Delta(s) \\ &= x + [\alpha a(s+\Delta, X_{s+\Delta}) + (1-\alpha)a]\Delta \\ &\quad +(1/2 - \alpha)[\beta L^0 a(s+\Delta, X_{s+\Delta}) + (1-\beta) L^0 a]\Delta^2 \\ &\quad +\alpha N_\Delta(s)\Delta + (1/2 - \alpha)\beta P_\Delta(s)\Delta^2 + M_\Delta(s). \end{aligned} \tag{3.27}$$

Implicit terms are contained in (3.27) up to Δ^2. The noise terms are taken from the rest according to whether they are needed to satisfy conditions for strong convergence of Ito approximation. In other words, for selected stochastic terms of the form:

$$\sum_{\alpha \in A_\gamma} g_\alpha I_\alpha,$$

require:

$$E(|g_\alpha - f_\alpha(Y_s)|^2) \leq K\delta^{2\gamma - \varphi(\alpha)}. \tag{3.28}$$

For the terms in the remainder, require cumulative error:

$$E(\max_{1 \leq n \leq n_T} | \sum_{0 \leq k \leq n-1} R_k|^2) \leq K\delta^{2\gamma}. \tag{3.29}$$

Example : $\alpha = 1$ for implicit Milstein. Also $\alpha = 1/2$ for implicit order 1.5 strong scheme. (β in this case can be anything.)