Lecture 9: Binomial random variables

Statistics 251

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Where are we?

Bernoulli and binomial random variables

Expectation and variance

Problems

Binomial random variables

Toss a fair coin n times. What is the probability of k heads?

What if the coin has probability p of coming up heads?

Binomial random variables

A **Bernoulli** random variable with parameter $p \in [0, 1]$ has value

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p. \end{cases}$$

A **binomial** random variable with parameters n, p has value

$$X = k \text{ with probability } \binom{n}{k} p^k (1-p)^{n-k}.$$

If $X_i \sim \text{Bernoulli}(p)$ for $i = 1, \ldots, n$, then

$$X_1 + X_2 + \cdots + X_n \sim \text{Binomial}(n, p).$$

Examples

If a room contains n people, what is the probability that exactly k of them were born on Monday?

If $n = 100$, what is the probability that at most 98 of them were born on Monday?

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Expectation

If $X \sim \text{Binomial}(n, p)$, what is $\mathbb{E}[X]$? We may compute

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{n} k\mathbb{P}(X = k) \\
&= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} k \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k} \\
&= np \sum_{k=0}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k} \\
&= np.
\end{aligned}
$$

Expectation (alternate approach)

If $X \sim \text{Binomial}(n, p)$, what is $\mathbb{E}[X]$?

Recall $X = X_1 + \cdots + X_n$, where $X_i \sim \text{Bernoulli}(n, p)$.

By linearity of expectation, we have

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X_1 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \\ &= np, \end{aligned}$$

where we note that $\mathbb{E}[X_i] = p$.

Again, again and again, in probabilty world, definition of independent and expectation do not always follow your intuition.

Variance

If $X \sim \text{Binomial}(n, p)$, what is $\text{Var}(X)$? We may compute

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{n} k^2 \mathbb{P}(X = k) \\
&= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} k^2 \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= np \sum_{k=0}^{n} k \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=0}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k} \\
&\quad + n(n-1)p^2 \sum_{k=0}^{n} \frac{(n-1)!}{(n-k)!(k-2)!} p^{k-2} (1-p)^{n-k}
\end{aligned}$$

Variance

If $X \sim \text{Binomial}(n, p)$, what is $\text{Var}(X)$? We may compute

$$\begin{aligned}
\mathbb{E}[X^2] &= np \sum_{k=0}^{n} k \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1}(1-p)^{n-k} \\
&= np \sum_{k=0}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1}(1-p)^{n-k} \\
&\qquad + n(n-1)p^2 \sum_{k=0}^{n} \frac{(n-2)!}{(n-k)!(k-2)!} p^{k-2}(1-p)^{n-k} \\
&= np + n(n-1)p^2 \\
&= np + n^2p^2 - np^2.
\end{aligned}$$

In addition, we have $\mathbb{E}[X]^2 = n^2p^2$, so

$$\text{Var}(X) = np + n^2p^2 - np^2 - n^2p^2 = np(1-p).$$

Variance (alternate approach)

If $X \sim \text{Binomial}(n, p)$, what is $\text{Var}[X]$?

Recall $X = X_1 + \cdots + X_n$, where $X_i \sim \text{Bernoulli}(n, p)$.

We may compute

$$\begin{aligned}\mathbb{E}[X^2] = \mathbb{E}\left[\sum_{i,j=1}^{n} X_i X_j\right] &= \sum_{i,j=1}^{n} \mathbb{E}[X_i X_j] \\ &= \sum_{i=1}^{n} \mathbb{E}[X_i^2] + 2\sum_{i<j} \mathbb{E}[X_i X_j] = np + n(n-1)p^2.\end{aligned}$$

As before, this means

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = np(1-p).$$

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An airplane seats 198, but the airline has sold 200 tickets. Each person independently has a 0.05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

Problems

In a 100 person senate, 40 people always vote for the Republicans, 40 people always vote for the Democrats, and 20 people toss a coin to decide which way to vote. What is the probability that a given vote is tied?