

Lecture 10: Strong Approximation of Stochastic Integrals

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Abstract

Stratonovich Taylor expansion; strong approximation by Fourier series expansion.

1 Stratonovich Taylor Expansion

1.1 1D Case

Given X_t satisfies,

$$X_t = X_{t_0} + \int_{t_0}^t a(X_s) ds + \int_{t_0}^t b(X_s) dW_s, \quad (1.1)$$

we know it also can be re-written as,

$$X_t = X_{t_0} + \int_{t_0}^t \underline{a}(X_s) ds + \int_{t_0}^t b(X_s) \circ dW_s, \quad (1.2)$$

where

$$\underline{a} = a - \frac{1}{2}bb'. \quad (1.3)$$

Given function f ,

$$\begin{aligned} f(X_t) &= f(X_{t_0}) + \int_{t_0}^t \left(a \frac{\partial}{\partial x} f(X_s) + \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2} f(X_s) \right) ds + \int_{t_0}^t b \frac{\partial}{\partial x} f(X_s) dW_s \\ &= f(X_{t_0}) + \int_{t_0}^t \left(a - \frac{1}{2}bb' \right) \frac{\partial}{\partial x} f(X_s) ds + \int_{t_0}^t b \frac{\partial}{\partial x} f(X_s) \circ dW_s \\ &= f(X_{t_0}) + \int_{t_0}^t \underline{L}^0 f(X_s) ds + \int_{t_0}^t \underline{L}^1 f(X_s) \circ dW_s \end{aligned} \quad (1.4)$$

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where we note,

$$\underline{L}^0 = \underline{a} \frac{\partial}{\partial x} \quad (1.5)$$

$$\underline{L}^1 = b \frac{\partial}{\partial x}. \quad (1.6)$$

Now,

$$\begin{aligned} X_t &= X_{t_0} \\ &+ \int_{t_0}^t \left(\underline{a}(X_{t_0}) + \int_{t_0}^s \underline{L}^0 \underline{a}(X_z) dz + \int_{t_0}^s \underline{L}^1 \underline{a}(X_z) \circ dW_z \right) ds \\ &+ \int_{t_0}^t \left(b(X_{t_0}) + \int_{t_0}^s \underline{L}^0 b(X_z) dz + \int_{t_0}^s \underline{L}^1 b(X_z) \circ dW_z \right) \circ dW_s \\ &= X_{t_0} + \underline{a}(X_{t_0}) \int_{t_0}^t ds + b(X_{t_0}) \int_{t_0}^t 1 \circ dW_s + R \end{aligned} \quad (1.7)$$

with remainder,

$$\begin{aligned} R &= \int_{t_0}^t \int_{t_0}^s \underline{L}^0 \underline{a}(X_z) dz ds + \int_{t_0}^t \int_{t_0}^s \underline{L}^1 \underline{a}(X_z) \circ dW_z ds \\ &+ \int_{t_0}^t \int_{t_0}^s \underline{L}^0 b(X_z) dz \circ dW_s + \int_{t_0}^t \int_{t_0}^s \underline{L}^1 b(X_z) \circ dW_z \circ dW_s \end{aligned} \quad (1.8)$$

1.2 General Case

Consider,

$$X_t = X_{t_0} + \int_{t_0}^t a(s, X_s) ds + \sum_{j=1}^m \int_{t_0}^t b^j(s, X_s) dW_s^j \quad (1.9)$$

we have

$$X_t = X_{t_0} + \int_{t_0}^t \underline{a}(s, X_s) ds + \sum_{j=1}^m \int_{t_0}^t b^j(s, X_s) \circ dW_s^j \quad (1.10)$$

with

$$\underline{a}^i = a^i - \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^d b^{k,j} \frac{\partial b^{i,j}}{\partial x^k}. \quad (1.11)$$

If

$$\underline{L}^0 = \frac{\partial}{\partial t} + \sum_{k=1}^d \underline{a}^k \frac{\partial}{\partial x^k} \quad (1.12)$$

$$\underline{L}^j = \sum_{k=1}^d b^{k,j} \frac{\partial}{\partial x^k} \quad (1.13)$$

then

$$\underline{a} = a - \frac{1}{2} \sum_{j=1}^m \underline{L}^j b^j. \quad (1.14)$$

Multiple stochastic integral and coefficient functions can be defined iteratively,

$$J_\alpha[g(\cdot, X.)]_{\rho, \tau} = \begin{cases} g(\tau, X_\tau) & : l = 0 \\ \int_\rho^\tau J_{\alpha-}[g(\cdot, X.)]_{\rho, s} ds & : l \geq 1, j_l = 0 \\ \int_\rho^\tau J_{\alpha-}[g(\cdot, X.)]_{\rho, s} \circ dW_s^{j_1} & : l \geq 1, j_l \geq 1 \end{cases} \quad (1.15)$$

$$\underline{f}_\alpha = \begin{cases} f & : l = 0 \\ \underline{L}^{j_1} \underline{f}_{-\alpha} & : l \geq 1 \end{cases} \quad (1.16)$$

Ito-Taylor Expansion: Finally, given an hierarchical set \mathcal{A} ,

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}} J_\alpha [\underline{f}_\alpha(\rho, X_\rho)]_{\rho, \tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} J_\alpha [\underline{f}_\alpha(\cdot, X.)]_{\rho, \tau}. \quad (1.17)$$

Relationship with Ito Integral, given $g \equiv 1$

- For $l(\alpha) \in \{0, 1\}$

$$I_\alpha = J_\alpha. \quad (1.18)$$

- For $l(\alpha) = 2$,

$$I_\alpha = J_\alpha - \frac{1}{2} I_{\{j_1=j_2 \neq 0\}} J_{(0)}. \quad (1.19)$$

- For $l(\alpha) = 3$,

$$I_\alpha = J_\alpha - \frac{1}{2} (I_{\{j_1=j_2 \neq 0\}} J_{(0, j_3)} + I_{\{j_2=j_3 \neq 0\}} J_{(j_1, 0)}). \quad (1.20)$$

- For $l(\alpha) = 4$,

$$\begin{aligned} I_\alpha = & J_\alpha + \frac{1}{4} I_{\{j_1=j_2 \neq 0\}} I_{\{j_3=j_4 \neq 0\}} J_{(0,0)} \\ & - \frac{1}{2} (I_{\{j_1=j_2 \neq 0\}} J_{(0, j_3, j_4)} + I_{\{j_2=j_3 \neq 0\}} J_{(j_1, 0, j_4)} + I_{\{j_3=j_4 \neq 0\}} J_{(j_1, j_2, 0)}). \end{aligned} \quad (1.21)$$

Relation between Stratonovich Integrals

$$W_t^j J_{\alpha, t} = \sum_{i=0}^l J_{(j_1, \dots, j_i, j, j_{i+1}, \dots, j_l), t} \quad (1.22)$$

Sketch of Proof: By Ito formula of function like $f(X, Y) = XY$, similar with one in lecture 8, in which we show,

$$W_t^j I_{\alpha,t} = \sum_{i=0}^l I_{(j_1, \dots, j_i, j; j_{i+1}, \dots, j_l), t} + \sum_{i=1}^l \mathbf{1}_{\{j_i=j \neq 0\}} I_{(j_1, \dots, j_{i-1}, 0, j_{i+1}, \dots, j_l), t} \quad (1.23)$$

Now,

$$W_t^j J_{\alpha,t} = J_{(j),t} J_{\alpha,t} = \int_0^t J_{\alpha,s} \circ dJ_{(j),s} + \int_0^t J_{(j),s} J_{\alpha-,s} \circ dW_s^{j_l} \quad (1.24)$$

$$= J_{(j_1, \dots, j_l, j), t} + \int_0^t W_s^j J_{\alpha-,s} \circ dW_s^{j_l} + \mathbf{1}_{\{j_l=j \neq 0\}} J_{(j_1, \dots, j_{l-1}, 0), t} \quad (1.25)$$

2 Strong Approximation of Stochastic Integrals

Motivation: *Riemann-Stieltjes integrals with respect to such a process will converge to Stratonovich stochastic integrals rather than to Ito stochastic integrals.*

2.1 Differentiable path approximation to a Wiener process

Consider the Brownian bridge process formed from a given m-dimensional Wiener process $W_t = (W_t^1, \dots, W_t^m)$,

$$\left\{ W_t - \frac{t}{\Delta} W_{\Delta}, 0 \leq t \leq \Delta \right\}. \quad (2.26)$$

We pathwisely consider Fourier expansion of the process,

$$W_t^j - \frac{t}{\Delta} W_{\Delta}^j = \frac{1}{2} a_{j,0} + \sum_{r=1}^{\infty} \left(a_{j,r} \cos\left(\frac{2r\pi t}{\Delta}\right) + b_{j,r} \sin\left(\frac{2r\pi t}{\Delta}\right) \right) \quad (2.27)$$

with random coefficients

$$a_{j,r} = \frac{2}{\Delta} \int_0^{\Delta} \left(W_s^j - \frac{s}{\Delta} W_{\Delta}^j \right) \cos\left(\frac{2r\pi s}{\Delta}\right) ds, \quad (2.28)$$

$$b_{j,r} = \frac{2}{\Delta} \int_0^{\Delta} \left(W_s^j - \frac{s}{\Delta} W_{\Delta}^j \right) \sin\left(\frac{2r\pi s}{\Delta}\right) ds, \quad (2.29)$$

and by setting $t = \Delta$,

$$a_{j,0} = -2 \sum_{r=1}^{\infty} a_{j,r}. \quad (2.30)$$

It can be shown that $a_{j,r}$ and $b_{j,r}$ are $N(0; \Delta/(2\pi^2 r^2))$ distributed and pairwise independent.

So we approximate W_t by,

$$W_t^{j,p} = \frac{t}{\Delta} W_\Delta^j + \frac{1}{2} a_{j,0} + \sum_{r=1}^p \left(a_{j,r} \cos\left(\frac{2r\pi t}{\Delta}\right) + b_{j,r} \sin\left(\frac{2r\pi t}{\Delta}\right) \right). \quad (2.31)$$

2.2 Representation of Stratonovich Stochastic Integrals

We write $\gamma = \frac{\pi}{\Delta}$, from definition,

$$J_{(0),t} = t, \quad J_{(0,0),t} = \frac{1}{2} t^2 \quad (2.32)$$

$$J_{(j),t} = \frac{1}{\Delta} W_\Delta^j J_{(0),t} + \frac{1}{2} a_{j,0} + \sum_{r=1}^{\infty} (a_{j,r} \cos(2\gamma r t) + b_{j,r} \sin(2\gamma r t)). \quad (2.33)$$

Then by integrating (2.31),

$$\begin{aligned} J_{(j,0),t} &= \int_0^t J_{(j),s} ds \\ &= \frac{1}{\Delta} W_\Delta^j J_{(0,0),t} + \frac{1}{2} a_{j,0} J_{(0),t} + \frac{\Delta}{2\pi} \sum_{r=1}^{\infty} \frac{1}{r} (a_{j,r} \sin(2\gamma r t) - b_{j,r} [\cos(2\gamma r t) - 1]). \end{aligned} \quad (2.34)$$

So, by setting $t = \Delta$,

$$J_{(0)} = \Delta, \quad J_{(j)} = W_\Delta^j, \quad J_{(0,0)} = \frac{1}{2} \Delta^2, \quad (2.35)$$

$$J_{(j,0)} = \frac{1}{2} \Delta (W_\Delta^j + a_{j,0}). \quad (2.36)$$

In addition, by (1.22),

$$J_{(0,j),\Delta} = J_{(j),t} J_{(0),\Delta} - J_{(j,0),\Delta} = \frac{1}{2} \Delta (W_\Delta^j - a_{j,0}). \quad (2.37)$$

Furthermore,

$$J_{(j_1,j_2)} = \int_0^\Delta J_{(j_1),s} \circ dW_s^{j_2} \quad (2.38)$$

$$= \frac{1}{\Delta} W_\Delta^{j_1} J_{(0,j_2)} + \frac{1}{2} a_{j_1,0} J_{(j_2)} + \int_0^\Delta \sum_{r=1}^{\infty} (a_{j_1,r} \cos(2\gamma r s) + b_{j_1,r} \sin(2\gamma r s)) \circ dW_s^{j_2} \quad (2.39)$$

$$= \frac{1}{2} W_\Delta^{j_1} W_\Delta^{j_2} - \frac{1}{2} (a_{j_2,0} W_\Delta^{j_1} - a_{j_1,0} W_\Delta^{j_2}) + \Delta A_{j_1,j_2}, \quad (2.40)$$

where, (note: $\cos^2 x = \frac{1+\cos 2x}{2}$)

$$A_{j_1, j_2} = \frac{1}{\Delta} \int_0^\Delta \sum_{r=1}^{\infty} (a_{j_1, r} \cos(2\gamma r s) + b_{j_1, r} \sin(2\gamma r s)) \circ dW_s^{j_2} \quad (2.41)$$

$$= \frac{1}{\Delta} \int_0^\Delta \left(\sum_{r=1}^{\infty} (a_{j_1, r} \cos(2\gamma r s) + b_{j_1, r} \sin(2\gamma r s)) \right) \quad (2.42)$$

$$\cdot \left(\frac{1}{\Delta} W_\Delta^{j_2} + 2\gamma \sum_{r=1}^{\infty} r (-a_{j_2, r} \sin(2\gamma r s) + b_{j_2, r} \cos(2\gamma r s)) \right) ds \quad (2.43)$$

$$= \frac{\pi}{\Delta} \sum_{r=1}^{\infty} r (a_{j_1, r} b_{j_2, r} - b_{j_1, r} a_{j_2, r}). \quad (2.44)$$

Based on this idea, we can derive,

$$J_{(0,0,0)} = \frac{1}{3!} \Delta^3, \quad J_{(0,j,0)} = \frac{1}{3!} \Delta^2 W_\Delta^j - \frac{1}{\pi} \Delta^2 b_j \quad (2.45)$$

$$J_{(j,0,0)} = \frac{1}{3!} \Delta^2 W_\Delta^j + \frac{1}{4} \Delta^2 a_{j,0} + \frac{1}{2\pi} \Delta^2 b_j \quad (2.46)$$

$$J_{(0,0,j)} = \frac{1}{3!} \Delta^2 W_\Delta^j - \frac{1}{4} \Delta^2 a_{j,0} + \frac{1}{2\pi} \Delta^2 b_j \quad (2.47)$$

with

$$b_j = \sum_{r=1}^{\infty} \frac{1}{r} b_{j,r} \quad (2.48)$$

$$J_{(j_1,0,j_2)} = \frac{1}{3!} \Delta W_\Delta^{j_1} W_\Delta^{j_2} + \frac{1}{2} a_{j_1,0} J_{(0,j_2)} + \frac{1}{2\pi} \Delta W_\Delta^{j_2} b_{j_1} - \Delta^2 B_{j_1, j_2} - \frac{1}{4} \Delta a_{j_2,0} W_\Delta^{j_1} + \frac{1}{2\pi} \Delta W_\Delta^{j_1} b_{j_2} \quad (2.49)$$

$$J_{(0,j_1,j_2)} = \frac{1}{3!} \Delta W_\Delta^{j_1} W_\Delta^{j_2} - \frac{1}{\pi} \Delta W_\Delta^{j_2} b_{j_1} + \Delta^2 B_{j_1, j_2} - \frac{1}{4} \Delta a_{j_2,0} W_\Delta^{j_1} + \frac{1}{2\pi} \Delta W_\Delta^{j_1} b_{j_2} + \Delta^2 C_{j_1, j_2} + \frac{1}{2} \Delta^2 A_{j_1, j_2} \quad (2.50)$$

with

$$B_{j_1, j_2} = \frac{1}{2\Delta} \sum_{r=1}^{\infty} (a_{j_1, r} a_{j_2, r} + b_{j_1, r} b_{j_2, r}) \quad (2.51)$$

and

$$C_{j_1, j_2} = -\frac{1}{\Delta} \sum_{\substack{r, l=1 \\ r \neq l}}^{\infty} \frac{r}{r^2 - l^2} (r a_{j_1, r} a_{j_2, l} + l b_{j_1, r} b_{j_2, l}) \quad (2.52)$$

$$J_{(j_1, j_2, 0)} = J_{(j_1, j_2)} - J_{(j_1, 0, j_2)} - J_{(0, j_1, j_2)} \quad (2.53)$$

2.3 Approximation

We using independent standard Gaussian random variables to represent,

$$\xi_j = \frac{1}{\sqrt{\Delta}} W_{\Delta}^j, \quad \zeta_{j,r} = \sqrt{\frac{2}{\Delta}} \pi r a_{j,r}, \quad \eta_{j,r} = \sqrt{\frac{2}{\Delta}} \pi r b_{j,r} \quad (2.54)$$

$$\mu_{j,p} = \frac{1}{\sqrt{\Delta} \rho_p} \sum_{r=p+1}^{\infty} a_{j,r}, \quad \phi_{j,p} = \frac{1}{\sqrt{\Delta} \alpha_p} \sum_{r=p+1}^{\infty} \frac{1}{r} b_{j,r} \quad (2.55)$$

where

$$\rho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^p \frac{1}{r^2}, \quad \alpha_p = \frac{\pi^2}{180} - \frac{1}{2\pi^2} \sum_{r=1}^p \frac{1}{r^4}. \quad (2.56)$$

Here p decides the truncated order and is set to be large enough.

For example,

$$J_{(0)}^p = \Delta, \quad J_{(j)}^p = \sqrt{\Delta} \xi_j, \quad J_{(0,0)}^p = \frac{1}{2} \Delta^2 \quad (2.57)$$

$$J_{(j,0)}^p = \frac{1}{2} \Delta \left(\sqrt{\Delta} \xi_j + a_{j,0} \right), \quad J_{(0,j)}^p = \frac{1}{2} \Delta \left(\sqrt{\Delta} \xi_j - a_{j,0} \right) \quad (2.58)$$

where

$$a_{j,0} = -\frac{1}{\pi} \sqrt{2\Delta} \sum_{r=1}^p \frac{1}{r} \zeta_{j,r} - 2\sqrt{\Delta} \rho_p \mu_{j,p} \quad (2.59)$$

$$J_{(j_1, j_2)}^p = \frac{1}{2} \Delta \xi_{j_1} \xi_{j_2} - \frac{1}{2} \sqrt{\Delta} (a_{j_2,0} \xi_{j_1} - a_{j_1,0} \xi_{j_2}) + \Delta A_{j_1, j_2}^p, \quad (2.60)$$

with

$$A_{j_1, j_2}^p = \frac{1}{2\pi} \sum_{r=1}^p \frac{1}{r} (\zeta_{j_1, r} \eta_{j_2, r} - \eta_{j_1, r} \zeta_{j_2, r}). \quad (2.61)$$

3 Project IV: Due before lecture 13

IV-1: Given 1D process driven by m -dimension Wiener process,

$$dX = a dt + \sum_{j=1}^m b^j dW^j, \quad (3.62)$$

determine $\underline{f}_{(j_1, j_2, j_3, j_4)}$ and $f_{(j_1, j_2, j_3, j_4)}$ for $j_1, \dots, j_4 \in \{1, \dots, m\}$.

IV-2: Approximating $W_1^1 I_{(1,2)}[1]_{0,1}$

1. Find a representation of $I_{(1,2)}$ in terms of some random coefficients.
2. Calculate $E(W_1^1 I_{(1,2)}[1]_{0,1})$ based on a truncation in (1) with Monte-Carlo.
3. Calculate $E(W_1^1 I_{(1,2)}[1]_{0,1})$ by direct sampling paths of Wiener process and integrating along path.