# Lecture 16: Joint Distributions

## Statistics 251

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# Lecture Outline

Joint distributions

Independent random variables

# Where are we?

Joint distributions

Independent random variables

# Joint distribution for discrete r.v.

If $X$ and $Y$ assume values in $\{1, 2, ..., n\}$then we can view $A_{i,j} = \mathbb{P}\{X = i, Y = j\}$ as the entries of an $n \times n$ matrix.

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In general, when $X$ and $Y$ are jointly defined discrete random variables, we write$\mathbb{P}(x, y) = \mathbb{P}_{X,Y}(x, y) = \mathbb{P}\{X = x, Y = y\}$.

# Joint distribution for continuous r.v.

Similar to discrete version, if given random variables $X$ and $Y$, define $F(a, b) = P\{X \leq a, Y \leq b\}$,

how to define marginal cumulative distribution function?
$F_X(a) =$

$F_Y(b) =$

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We should first verify, for $A = \{(X, Y) : X \le a, Y \le b\}$,

$$\mathbb{P}\{(X, Y) \in A\} = \int_A f(x, y)dxdy$$

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Then we should show it works for strips, rectangles and general open sets.

# Example: From density to probability

The joint density function of $X$ and $Y$ is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute
$\mathbb{P}\{X > 1, Y < 1\}$

$\mathbb{P}\{X < Y\}$

$\mathbb{P}\{X < a\}$

# Example: Density of functions of r.v.

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What is the density function of random variable $\frac{X}{Y}$

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# Definition

We say $X$ and $Y$ are independent if for any two (measurable) events $A$ and $B$ we have

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What about density function if $X$ and $Y$ are continous?

# Example: Waiting time

A couple ($A$, $B$) decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 P.M., find the probability that the first to arrive has to wait longer than 10 minutes.

# Example: Buffon's needle problem

A table is ruled with equidistant parallel lines a distance $D$ apart. A needle of length $L$, where $L \leq D$, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)?

# Example: Convolution of distribution

Given $X$, $Y$ independent, continous r.v. with density function $f_X$, $f_Y$. what is probability density function of $X + Y$?