

# Lecture 3: Integral

Zhongjian Wang\*

## Abstract

Ito Integral, Ito Formula, Stratonovich Integral

## 1 Motivation and Definition

### 1.1 SDE of OU

Recall in last lecture, for OU process with coefficient  $\gamma$ ,  $\forall \tau \geq 0$ ,  $X(t + \tau) - e^{-\gamma t}X(\tau)$  is independent of  $(\omega : X(s), s \leq \tau)$ ,

Calculate drift:

$$E(X(t) - X(s)|X(s) = x) = (e^{-\gamma|t-s|} - 1)x, \quad (1.1)$$

$$a(s, x) = -\gamma x \quad (1.2)$$

Calculate diffusion:

$$E((X(t) - X(s))^2|X(s) = x) = 1 - e^{-2\gamma(t-s)} + (e^{-\gamma(t-s)} - 1)^2x^2 \quad (1.3)$$

$$b^2(s, x) = 2\gamma \quad (1.4)$$

BTW, jump definition

$$\lim_{t \rightarrow s^+} \frac{1}{t-s} \int_{|y-x| > \epsilon} p(s, x; t, y) dy = \lim_{t \rightarrow s^+} \frac{1}{t-s} P(|X(t) - X(s)| > \epsilon | X(s) = x) \quad (1.5)$$

$$\leq \lim_{t \rightarrow s^+} \frac{1}{t-s} \frac{E((X(t) - X(s))^2 | X(s) = x)}{\epsilon^2} = 0 \quad (1.6)$$

Over small time interval  $[s, t]$ , using drift-diffusion information, we see that O-U is related to BM as (to leading order):

$$X(t) - X(s) = -\gamma X(s)(t-s) + \sqrt{2\gamma}(W(t) - W(s)),$$

where  $W(t)$  denotes BM; or in differential form:

$$dX = -\gamma X dt + \sqrt{2\gamma} dW.$$

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\*Department of Statistics, University of Chicago

## 1.2 Integral Form of SDE

For general SDE of the form:

$$dX_t = a(t, X_t) dt + b(t, X_t) dW_t,$$

the integral equation is:

$$\begin{aligned} X_t(\omega) = X(t, \omega) &= X_{t_0}(\omega) + \int_{t_0}^t a(s, X_s(\omega)) ds \\ &+ \int_{t_0}^t b(s, X_s(\omega)) dW_s(\omega). \end{aligned} \quad (1.7)$$

$dW_s$  is not a regular function, if we define  $dW_t = \xi_t dt$ ,  $\xi_t$  white noise, it need some definition either on the integrability. So we consider in general:

$$I(f)(\omega) = \int_0^1 f(s, \omega) dW_s(\omega),$$

$f$  is a random function of  $t$  (stochastic process).

Naturally, one approximates:

$$I(f)(\omega) \approx \sum_{j=1}^n f_j [W_{t_{j+1}} - W_{t_j}],$$

for a partition  $0 = t_1 < t_2 < \dots < t_{n+1} = 1$ .

*The issue is how to choose  $f_j$ .* As both  $f_j$  and increment of BM are random, the product is difficult to handle in general ("coupled" or "correlated") unless there is some way to introduce *decoupling* !

*Adapted (a.k.a. nonanticipating)  $f$ :* let  $A_t$  be a sequence of increasing  $\sigma$ -algebra such that  $W_t$  is measurable for each  $t > 0$ , or  $A_t$  contains all the events where we observe BM up to time  $t$ .

Suppose  $f$  is a step function:

$$f(t, \omega) = f_j(\omega), \quad t \in [t_j, t_{j+1}].$$

We call  $f$  adapted if  $f_j$ 's are all  $A_{t_j}$  measurable or observable by events at or before time  $t_j$ . If  $f$  is not step function, define  $f$  as a limit (in the mean square sense) of such adapted step functions.

With such additional information on  $f_j$ , we can take  $E[\cdot]$ :

$$\begin{aligned} E(f_j [W_{t_{j+1}} - W_{t_j}]) &= E(E(f_j [W_{t_{j+1}} - W_{t_j}] | A_{t_j})) \\ &= E(f_j E([W_{t_{j+1}} - W_{t_j}] | A_{t_j})) = 0. \end{aligned} \quad (1.8)$$

Also:

$$\begin{aligned}
E(I(f)^2) &= \sum_{j=1}^n E(f_j^2 E([W_{t_{j+1}} - W_{t_j}]^2 | \mathcal{A}_{t_j})) \\
&= \sum_{j=1}^n E(f_j^2) (t_{j+1} - t_j),
\end{aligned} \tag{1.9}$$

which is a Riemann sum.

If  $f$  is a step function,  $E(I(f)^2)$  is independent of time partition  $\gamma = t_1, \dots, t_n$ , given  $f$  is constant in each interval of  $\gamma$

If  $f$  is not a step function, suppose  $E(|f_n - f|^2) \rightarrow 0$ , Ito **showed** that  $I(f_n)$  has a unique limit, i.e. independent of time partition, defined as  $I(f)$ , the Ito stochastic integral.

Properties:

- $I(f)$  is  $\mathcal{A}_1$  measurable, (1 is end time of integral), so it is a **r.v. in  $L^2(\Omega, \mathcal{A}_1, P)$** ;
- $E(I(f)) = 0$ ,  $E(I(f)^2) = \int_0^1 E(f(s, \omega))^2 ds$ .
- $I(f)$  is linear in  $f$ .

**Stochastic process:** Now we de-freeze  $t$  consider an Ito integral defined on  $[0, t]$ :

$$X_t(\omega) = \int_0^t f(s, \omega) dW_s(\omega),$$

is  $\mathcal{A}_t$  measurable, adapted, mean zero, and:

$$E(X_t^2) = \int_0^t E(f(s, \omega)^2) ds.$$

So  $\int_0^t f(s, \omega) dW_s(\omega)$  is a stochastic process.

### 1.3 Sketch of Proof by Ito

Define the space of stochastic process ( $\mathcal{L}_T^2$ ) by norm,

$$\|f\|_{2,T} = \sqrt{\int_0^T E(f(t, \cdot)^2) dt}. \tag{1.10}$$

We denote by  $\mathcal{S}_T^2$  the subset of all step functions in  $\mathcal{L}_T^2$ . Then we can approximate any function in  $\mathcal{L}_T^2$  by step functions in  $\mathcal{S}_T^2$  to any desired degree of accuracy in the norm. To be specific we have

$\mathcal{S}_T^2$  is dense in  $(\mathcal{L}_T^2, \|\cdot\|_{2,T})$ ,  $I(f)$  in  $\mathcal{S}_T^2$  is well-defined and satisfies the properties.

Next, for an arbitrary function  $f \in \mathcal{L}_T^2$ , let  $f^{(n)} \in \mathcal{S}_T^2$  a sequence of step functions that,

$$\int_0^T E(|f^{(n)}(t, \cdot) - f(t, \cdot)|^2) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

The last step, we will prove  $I(f^{(n)})$  is a Cauchy sequence in the Banach space  $L^2(\Omega, \mathcal{A}_T, P)$ .

## 2 Properties of Ito Integrals

### 2.1 Martingale

We know the Ito integral  $I(f)$  is well-defined by passing to limit an approximation:

$$I(f) = \sum_{j=1}^n f(t_j, \omega)[W_{t_{j+1}}(\omega) - W_{t_j}(\omega)],$$

the left-hand rule. Such  $I(f)$  has nice properties:

- $I(f)$  is  $A_T$  measurable,  $E(I(f)) = 0$ ;
- $E(I(f)^2) = \|f\|_{2,T}^2$ ,  $I$  is linear in  $f$ .

It follows:

$$E(I(f)I(g)) = \int_0^T E(f(t, \cdot)g(t, \omega)) dt.$$

Define  $Z_t$  to be the Ito integral when  $T$  is set to  $t$ , then:

$$E(Z_t - Z_s | A_s) = 0, \tag{2.11}$$

for  $s \leq t$ . Such a process  $Z_t$  is called **martingale**. If  $=$  is replaced by  $\leq$  ( $\geq$ ), supermartingale (submartingale).

Example:  $\xi_n = 1$  if a tossed fair coin is head, otherwise  $-1$ . Let  $A_n = \sigma(\xi_1, \xi_2, \dots, \xi_n)$ ,  $n \geq 1$ ,  $X_0 = 0$ ,  $A_0 = \{\varphi, \Omega\}$ ,  $X_n = \xi_1 + \dots + \xi_n$  is a martingale with respect to  $A_n$ .

- Let  $g$  be any convex function, then  $g(X_t)$  is a submartingale:

$$E(g(X_t) | A_s) \geq g(E(X_t | A_s)) = g(X_s), \quad k \leq n.$$

by Jensen's inequality.

### 2.2 Inequalities

Given  $Z_t$  is a mean-square martingale, the maximal martingale inequality holds,

$$P\left(\sup_{t_0 \leq s \leq t} |Z_s| \geq a\right) \leq \frac{1}{a^2} \int_{t_0}^t E(f(s, \cdot)^2) ds,$$

for any  $a > 0$ , and the Doob inequality holds

$$E\left(\sup_{t_0 \leq s \leq t} |Z_s|^2\right) \leq 4 \int_{t_0}^t E(f(s, \cdot)^2) ds.$$

### 2.3 Paths continuous (\*see page 87-88 KL's book)

Note  $I(f) - I(f^{(n)})$  is also martingale, and  $I(f^{(n)})$  are obviously continuous. Now by carefully select  $f^{(n)}$ , we can prove,

There is a separable, jointly measurable version of  $Z_t$  defined by

$$Z_t(\omega) = \int_{t_0}^t f(s, \omega) dW_s(\omega)$$

for  $t \in [t_0, T]$  has, **almost surely, continuous sample paths.**

## 3 Peculiarities of Ito Integrals

Ito integral is different from Riemann integral in your calculus textbook:

$$\int_0^t W_s(\omega) dW_s(\omega) \neq W_t^2(t, \omega)/2,$$

instead:

$$\int_0^t W_s(\omega) dW_s(\omega) = W_t^2(t, \omega)/2 - t/2. \tag{3.12}$$

$$\sum_{j=1}^n W_{t_j} (W_{t_{j+1}} - W_{t_j}) = W_t^2/2 - \frac{1}{2} \sum_{j=1}^n ((W_{t_{j+1}} - W_{t_j})^2).$$

The last sum is over i.i.d random variable, hence converging to its mean w.p. 1, that is  $t$ .

Next consider chain rule for a composite function  $Y_t = U(t, X_t)$ , where  $dX_t = b dW_t$ . The difference is Taylor expanded:

$$\begin{aligned} \Delta Y_t &= U(t + \Delta t, X_t + \Delta X_t) - U(t, X_t) \\ &= U_t \Delta t + U_X \Delta X_t + \\ &\quad + \frac{1}{2} [U_{tt} (\Delta t)^2 + 2U_{tX} \Delta t \Delta X_t + U_{XX} (\Delta X_t)^2]. \end{aligned} \tag{3.13}$$

The second order terms go to zero if  $X$  is a smooth function, however now  $E[(\Delta X_t)^2] \approx E[b^2] \Delta t$ , behaving like first order term  $U_t \Delta t$ . *It is this correction that makes the difference.*

*Ito formula:*

$$dY_t = (U_t + \frac{1}{2} b^2 U_{XX}) dt + U_X dX_t. \tag{3.14}$$

## 4 General Ito's Formula

The price to pay for having the nice properties of last section is a different rule for calculus. Let:

$$dX_t = a(t, \omega) dt + b(t, \omega) dW_t,$$

both  $e$  and  $f$  adapted; let  $U = U(t, x)$  be  $C^1$  in  $t$ ,  $C^2$  in  $x$ . Then:

$$\begin{aligned}
 U(t, X_t) &= U(s, X_s) + \int_0^t [U_t + aU_x + b^2U_{xx}/2](s, X_s) ds \\
 &\quad + \int_0^t bU_x(s, X_s) dW_s,
 \end{aligned} \tag{4.15}$$

The operator appearing in the first integral:

$$LU \equiv aU_x + b^2U_{xx}/2,$$

is dual of the right hand side of Kolmogorov forward equation for transitional probability density, recalling that

$$p_t = -(a(t, x)p)_x - \frac{1}{2}(b^2(t, x)p)_{xx},$$

## 5 Stratonovich Integral

Instead of left hand rule, one could also consider the integral being approximated by:

$$S_\lambda(f) = \sum_j [(1 - \lambda)f_j + \lambda f_{j+1}][W_{j+1} - W_j],$$

where  $\lambda \in [0, 1]$ ,  $W_j = W(t_j, \omega)$ . The resulting integrals, converge and are so called  $(\lambda)$ -integrals.

Example:

$$(\lambda) \int_0^T W_t dW_t = W_T^2/2 + (\lambda - \frac{1}{2})T.$$

Write  $S_\lambda(W_t)$  into two parts:

$$\begin{aligned}
 \sum_j W_j[W_{j+1} - W_j] &\rightarrow W_T^2/2 - T/2, \\
 \sum_j W_{j+1}[W_{j+1} - W_j] &= \sum_j (W_{j+1} - W_j)^2 + W_j[W_{j+1} - W_j] \\
 &\rightarrow T + W_T^2/2 - T/2.
 \end{aligned} \tag{5.16}$$

The mid point rule  $\lambda = 1/2$  is called Stratonovich, where BM correction vanishes, and standard calculus applies.

Relationship between Ito and Stratonovich ( $f = f(W_t)$ ):

$$\begin{aligned}
S_{1/2}(f) &= \sum_j f(W_j)[W_{j+1} - W_j] \\
&+ \frac{1}{2} \sum_j f'(W_j)[W_{j+1} - W_j]^2 + \dots \\
&\rightarrow \int_0^T f(W_t) dW_t + \frac{1}{2} \int_0^T f'(W_t) dt,
\end{aligned} \tag{5.17}$$

in the mean square sense.

Stratonovich integral  $S_{1/2}(f)$  has special notation,

$$\int_0^T f(W_t) \circ dW_t = \int_0^T f(W_t) dW_t + \frac{1}{2} \int_0^T f'(W_t) dt. \tag{5.18}$$

Consider definite integral  $F$  such that  $F' = f$ , Ito formula gives:

$$F(W_t) - F(W_0) = \int_0^T f(W_t) dW_t + \frac{1}{2} \int_0^T f'(W_t) dt. \tag{5.19}$$

Hence:

$$\int_0^T f(W_t) \circ dW_t = F(W_t) - F(W_0),$$

as in classical calculus.

## 6 Stratonovich SDE

$$dX_t = a(t, X_t)dt + b(t, X_t) \circ dW_t, \tag{6.20}$$

or:

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) \circ dW_s, \tag{6.21}$$

where the stochastic integral is understood as mean square limit of:

$$S_n(\omega) = \sum_{j=0}^n b(t_j, (X_{t_j} + X_{t_{j+1}})/2)(W_{t_{j+1}} - W_{t_j}), \tag{6.22}$$

as  $k = t/n \rightarrow 0$ , where  $t_j = jk$ ,  $k = t/n$ ,  $j = 0, 1, \dots, n$ .

Consider Stratonovich integral:

$$\int_0^t h(s, X_s) \circ dW_s, \tag{6.23}$$

- Assume for any finite  $T$ :

$$\int_0^T E(|h(t, X_t)|^2) dt < \infty,$$

then

**Theorem 6.1 (I-S integral transform)**

$$\int_0^T h(t, X_t) \circ dW_t = \int_0^T h(t, X_t) dW_t + \frac{1}{2} b(t, X_t) h_x(t, X_t) dt. \quad (6.24)$$

*Sketch of Proof:* let  $X_j = X_{t_j}$ ,

$$\begin{aligned} & h(t_j, (X_j + X_{j+1})/2) - h(t_j, X_j) \\ &= \frac{1}{2} h_x(t_j, \frac{1}{2}((2 - \theta_j)X_j + \theta_j X_{j+1}))(X_{j+1} - X_j), \end{aligned}$$

for random numbers  $\theta_j$ 's.

$$\begin{aligned} \Delta X_j &= X_{j+1} - X_j \\ &= a(t_j, X_j) \Delta t + b(t_j, X_j) \Delta W_j + h.o.t, \end{aligned} \quad (6.25)$$

Each term in the Stratonovich sum is:

$$\begin{aligned} & h(t_j, X_j) \Delta W_j + \frac{1}{2} h_x(\theta_j) \Delta X_j \Delta W_j \\ &= h(t_j, X_j) \Delta W_j + \frac{1}{2} h_x(\theta_j) b(t_j, X_j) (\Delta W_j)^2 \\ & \quad + \frac{1}{2} h_x(\theta_j) a(t_j, X_j) \Delta t \Delta W_j, \end{aligned} \quad (6.26)$$

$$h_x(\theta_j) = h_x(t_j, \frac{1}{2}((2 - \theta_j)X_j + \theta_j X_{j+1})).$$

We derive (6.24) using  $E((\Delta W_j)^2) = \Delta t$ ,  $E(k \Delta W_j) = 0$ , and ignoring higher order terms. It follows from (6.24):

$$E\left[\int_0^T h(t, X_t) \circ dW_t\right] = \frac{1}{2} \int_0^T E[b(t, X_t) h_x(t, X_t)] dt.$$

**Theorem 6.2 (I-S SDE transform)** *Let  $X_t$  solve Ito SDE:*

$$dX_t = a(t, X_t) dt + b(t, X_t) dW_t,$$

*then  $X_t$  solves the Stratonovich SDE:*

$$dX_t = (a(t, X_t) - \frac{1}{2} b(t, X_t) b_x(t, X_t)) dt + b(t, X_t) \circ dW_t.$$



## 7 Application

Consider homogeneous linear Ito SDE:

$$dX_t = aX_t dt + bX_t dW_t,$$

$a, b$  constant.

S-T transform gives the Stratonovich SDE:

$$dX_t = (a - b^2/2)X_t dt + bX_t \circ dW_t,$$

solution:

$$X_t = X_s \exp\{(a - b^2/2)(t - s) + b(W_t - W_s)\}.$$