Lecture 7: Weak convergence, Numerical Stability

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Abstract
Weak consistency implies weak convergence; numerical stability.

1 Weak Consistency: Definition and Examples

A discrete SDE approximation $Y^\delta(t)$ is called *converging weakly* to $X(t)$ at $t = T$ if:

$$\lim_{\delta \to 0} |E(g(X(T))) - E(g(Y^\delta(T)))| = 0, \tag{1.1}$$

for any $g \in \mathcal{C}$, $\mathcal{C}$ a class of smooth test functions. One example of $\mathcal{C}$ is all polynomials, then (1.1) is same as convergence of all moments of solutions. As before, discrete times $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N = T$, $\Delta_n = t_{n+1} - t_n$, $\delta = \max \Delta_n$.
Convergence is order $\beta > 0$ if:

$$|E(g(X(T))) - E(g(Y^\delta(T)))| \le C\delta^\beta, \tag{1.2}$$

for small δ.
Later we will see that Euler method is weakly convergent of order $\beta = 1$, while it is order $1/2$ strong convergent (pathwise).

The discrete approximation is *weakly consistent* if

$$E\left(\left|E\left(\frac{Y^\delta_{n+1} - Y^\delta_n}{\Delta_n}|A_{t_n}\right) - a(t_n, Y^\delta_n)\right|^2\right) \le c(\delta) \to 0, \tag{1.3}$$

same as in strong consistency, and:

$$\begin{aligned} &E\left[\left|E\left(\frac{1}{\Delta_n}(Y^\delta_{n+1} - Y^\delta_n)^2|A_n\right) - b^2(t_n, Y^\delta_n)\right|^2\right] \\ &\le c(\delta) \to 0. \end{aligned} \tag{1.4}$$

for all fixed $Y^\delta_n = y$, $n = 0, 1, 2, \cdots$.
For Euler, weak consistency holds. Moreover, some modified Euler like:

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta_n + b(t_n, Y_n)\xi_n (\Delta_n)^{1/2}, \tag{1.5}$$

where ξ_n independent two point r.v., $P(\xi_n = \pm 1) = 1/2$, is weakly convergent, not strongly convergent.

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2 Consistency implies Convergence

Consider the autonomous SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{2.6}$$

a, b, smooth, with polynomial growth.

Theorem 2.1 *Consider equidistant time weakly consistent discrete approximation Y_n^δ of (2.6) with $Y^\delta(0) = X_0$ so that:*

$$E(\max_n |Y_n^\delta|^{2q}) \leq K(1 + E(|X_0|^{2q})), \tag{2.7}$$

for $q = 1, 2, \cdots$, and:

$$E(|Y_{n+1}^\delta - Y_n^\delta|^6) \leq c(\delta)\Delta_n, \quad c(\delta) = o(\delta), \tag{2.8}$$

for any $n = 0, 1, 2, \cdots$. Then Y_n^δ converges weakly to $X(t)$.

Sketch of Proof: Write $Y(t) = Y^\delta(t)$.
Use fact:

$$u(s, x) = E(g(X_T)|X_s = x), \tag{2.9}$$

solves backward equation:

$$u_s + Lu = u_s + au_x + \frac{b^2}{2}u_{xx} = 0, \tag{2.10}$$

and:

$$u(T, x) = g(x). \tag{2.11}$$

Denote by $X_t^{s,x}$ solution of:

$$X_t^{s,x} = x + \int_s^t a(X_r^{s,x})dr + \int_s^t b(X_r^{s,x})dW_r. \tag{2.12}$$

Ito formula and (2.10) give:

$$E(u(t_{n+1}, X_{t_{n+1}}^{t_n,x}) - u(t_n, x)|A_n) = 0, \tag{2.13}$$

By eqns (2.9)-(2.11), write:

$$\begin{aligned} H &= |E(g(Y(T))) - E(g(X(T)))| \\ &= |E(u(T, Y(T)) - u(0, Y_0))| \\ &= |E(\sum_{n=0}^{n_T-1} u(t_{n+1}, Y_{n+1}) - u(t_n, Y_n))|. \end{aligned} \tag{2.14}$$

By (2.13):

$$\begin{aligned} H &= |E(\sum[u(t_{n+1}, Y_{n+1}) - u(t_n, Y_n) \\ &\quad -(u(t_{n+1}, X_{t_{n+1}}^{t_n,Y_n}) - u(t_n, X_{t_n}^{t_n,Y_n}))])| \\ &= |E(\sum[u(t_{n+1}, Y_{n+1}) - u(t_{n+1}, Y_n) \\ &\quad -(u(t_{n+1}, X_{t_{n+1}}^{t_n,Y_n}) - u(t_{n+1}, Y_n))])| \end{aligned}$$

Taylor expand in x:

$$\begin{aligned} H &= |E(\sum u_x[(Y_{n+1} - Y_n) - (X_{t_{n+1}}^{t_n,Y_n} - Y_n)] \\ &\quad +\frac{1}{2}u_{xx}[(Y_{n+1} - Y_n)^2 - (X_{t_{n+1}}^{t_n,Y_n} - Y_n)^2] \\ &\quad +O(|Y_{n+1} - Y_n|^3 + |X_{t_{n+1}}^{t_n,Y_n} - Y_n|^3))| \end{aligned} \tag{2.15}$$

u_x, u_{xx} evaluated at (t_{n+1}, Y_n).

Higher Moments Estimate of SDE (augmented, Theorem 4.5.4 in KL's book)
Suppose that conditions in lecture 5 hold and that

$$E\left(|X_{t_0}|^{2n}\right) < \infty$$

for some integer $n \geq 1$. Then the solution X_t satisfies

$$E\left(|X_t|^{2n}\right) \leq \left(1 + E\left(|X_{t_0}|^{2n}\right)\right) e^{C(t-t_0)}$$

and

$$E\left(|X_t - X_{t_0}|^{2n}\right) \leq D\left(1 + E\left(|X_{t_0}|^{2n}\right)\right)(t - t_0)^n e^{C(t-t_0)}$$

$$\begin{aligned} H &\leq C \sum E(|u_x||E((Y_{n+1} - Y_n) - (X_{t_{n+1}}^{t_n,Y_n} - Y_n)|A_n)| \\ &+ \frac{1}{2}|u_{xx}||E((Y_{n+1} - Y_n)^2 - (X_{t_{n+1}}^{t_n,Y_n} - Y_n)^2|A_n)| \\ &\quad +O(\delta^{3/2} + \delta^{1/2}\sqrt{c(\delta)}) \\ &\leq C\delta \sum E^{1/2}(|E(\frac{Y_{n+1} - Y_n}{\delta}|A_n) - a(t_n, Y_n)|^2) \\ &+ E^{1/2}(|E(\frac{(Y_{n+1} - Y_n)^2}{\delta}|A_n) - b^2(t_n, Y_n)|^2) \\ &+ O(\delta^{3/2} + \delta^{1/2}\sqrt{c(\delta)}) \\ &\leq C \sum \delta\sqrt{c(\delta)} + O(\delta^{3/2} + \delta^{1/2}\sqrt{c(\delta)}) \\ &= O(\sqrt{c(\delta)} + \delta^{1/2} + \sqrt{c(\delta)/\delta}) \to 0. \end{aligned} \tag{2.16}$$

3 Numerical Stability and A-Stability

A discrete approximation Y^δ of Ito SDE is stable if for two initial data Y_0^δ and $\tilde{Y}_0^\delta$:

$$\lim_{|Y_0^\delta - \tilde{Y}_0^\delta| \to 0} \sup_{t\in[0,T]} P(|Y_t^\delta - \tilde{Y}_t^\delta| \geq \epsilon) = 0, \tag{3.17}$$

for each $\epsilon > 0$, $\delta \in (0, \delta_0)$, $\delta_0 > 0$.
For the Euler method, following the same estimates as in uniqueness proof, we derive:

$$Z_t = \sup_{s\in[0,t]} E(|Y_s^\delta - \tilde{Y}_s^\delta|^2) \leq |Y_0^\delta - \tilde{Y}_0^\delta|^2 + C\int_0^t Z_s ds, \tag{3.18}$$

Gronwall inequality implies:

$$Z_t \leq |Y_0^\delta - \tilde{Y}_0^\delta|^2 \,(1 + e^{C_1 T}), \tag{3.19}$$

hence (3.17). Stability only refers to closeness of solutions on a finite interval $[0, T]$ for small enough time step δ.
Asymptotic stability extends stability to $T = \infty$ as:

$$\lim_{|Y_0^\delta - \tilde{Y}_0^\delta| \to 0} \lim_{T\to\infty} P(\sup_{t\in[0,T]} |Y_t^\delta - \tilde{Y}_t^\delta| \geq \epsilon) = 0. \tag{3.20}$$

To help determine asymptotic stability, consider the test eqn:

$$dX_t = \lambda X_t dt + dW_t, \tag{3.21}$$

where $Re(\lambda) < 0$. Applying a discretization method to (3.21) gives:

$$Y_{n+1} = G(\lambda\delta) Y_n + Z_n, \tag{3.22}$$

Z_n are r.v independent of λ, and Y_n.
Region of absolute stability is:

$$\{\lambda\delta \in \mathcal{C} : Re(\lambda) < 0,\ |G(\lambda\delta)| < 1\}, \tag{3.23}$$

Example 1: Euler method:

$$Y_{n+1} = Y_n(1 + \lambda\delta) + W_{n+1} - W_n, \tag{3.24}$$

$$|Y_{n+1} - \tilde{Y}_{n+1}| \leq |1 + \lambda\delta||Y_n - \tilde{Y}_n|$$

absolute-stable if:

$$|1 + \lambda\delta| < 1,\ Re(\lambda) < 0.$$

Example 2: Implicit Euler method:

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\delta + b(t_n, Y_n)(W_{n+1} - W_n), \tag{3.25}$$

takes the form on eqn (3.21):

$$Y_{n+1} = Y_n + Y_{n+1}\lambda\delta + W_{n+1} - W_n,$$

so:

$$|Y_{n+1} - \tilde{Y}_{n+1}| \leq |1 - \lambda\delta|^{-1}|Y_n - \tilde{Y}_n|,$$

absolute-stable for all $Re(\lambda) < 0$, any step size δ, i.e. absolute stable in the left half plane, which is called **A-stable**.

For cases of multiplicative noise, we applied fully implicit Euler method:

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\delta + b(t_{n+1}, Y_{n+1})(W_{n+1} - W_n), \tag{3.26}$$

applied to

$$dX_t = \lambda X_t dt + X_t dW_t,$$

yields:

$$Y_n = Y_0 \prod_{k=0}^{n-1} \frac{1}{1 - \lambda\delta - (W_{k+1} - W_k)}, \tag{3.27}$$

is not suitable for strong approximation as the denominator can be zero. It is fine for weak approximation with iid two point process U_k replacing $W_{k+1} - W_k$:

$$P(U_k = \pm\sqrt{\delta}) = 1/2.$$

4 Project III (due before lecture 10)

III1. Consider the SDE:

$$dX_t = aX_t dt + bX_t dW_t,$$

a, b constants, and its Euler scheme. Find the order(s) of convergence of the third and fourth moments of the approximate solutions.

III2. Consider the initial boundary value problem of:

$$u_t = 0.025\, u_{xx} + e^{\xi(x,\omega)} u(1-u), \quad x \in [0, 15],$$

$\xi(x,\omega)$ is the stationary O-U process with $N(0,1)$ at $x = 0$, covariance $E(\xi(x)\xi(0)) = e^{-2x}$ as in Project II. Use backward-time-central-space scheme with a proper h, k to discretize the SPDE, $h \leq 0.01$. Boundary conditions are: $u(t,0) = 1$, $u(t,15) = 0$; and initial condition: $u(0,x) = \chi_{[0,1]}(x)$. Evolve numerically to $t = 20$.

(1) Plot a sample solution u for $t = 0, 4, 8, 12, 16, 20$. (You should see some propogating front profile)

(2) Generate $N \geq 1000$ samples. For each ensemble solution $u(\cdot, \cdot; \omega)$, we define a random process, $X(t,\omega)$ such that, $u(X(t,\omega), t; \omega) = 1/2$. Plot a histogram of $\eta_1(\omega) = X(20,\omega)/20$.

(3) Calculate $c = E(\eta_1)$, and

$$c' = 2\sqrt{0.025 * E(e^{\xi})},$$

the latter being the naive estimate of random front velocity. Which average speed is larger ?

III3. The SDE:

$$dX_t = aX_t dt + bX_t dW_t,$$

a, b constants, has exact solution:

$$X_t = X_0 \exp\{(a - \frac{b^2}{2})t + bW_t\}.$$

Let $X_0 = 1$, $a = 1.5$, $b = 1$. Solve the SDE for $t \in [0,1]$ numerically by Euler and Milstein (search *Milstein method* on Wikipedia) schemes, with time step $\delta = 2^{-n}$, $n = 3, 4, 5, 6$.

(1) Plot a sample solution computed with Euler and Milstein, together with exact solution, for the above δ's;

(2) generate $20,000$ samples for each value of δ, and compute the absolute error $\epsilon = \epsilon(\delta) = E(|X(1) - Y^{\delta}(1)|)$. Plot ϵ vs. δ, $\delta = 2^{-n}$, $n = 3, 4, 5, 6$. Conclude on the order of accuracy of Euler and Milstein.