Lecture 4: Solvable SDEs

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Abstract

linear SDEs, O-U, Solutions and Moments, reducible SDEs.

1 Vector Valued Ito Integral

We write symbolically as a d -dimensional vector stochastic differential

$$dX_t = e_t dt + F_t dW_t. \tag{1.1}$$

Then for any $0 \leq s \leq t \leq T$, which we interpret componentwise as

$$X_t^k - X_s^k = \int_s^t e_u^k du + \sum_{j=1}^m \int_s^t F_u^{k,j} dW_u^j.$$

We define a scalar process $\{Y_t, 0 \leq t \leq T\}$ by

$$Y_t = U(t, X_t) = U\left(t, X_t^1, X_t^2, \ldots, X_t^d\right)$$

Then the stochastic differential for Y_t is given by

$$\begin{aligned} dY_t = &\left\{ \frac{\partial U}{\partial t} + \sum_{k=1}^d e_t^k \frac{\partial U}{\partial x_k} + \frac{1}{2} \sum_{j=1}^m \sum_{i,k=1}^d F_t^{i,j} F_t^{k,j} \frac{\partial^2 U}{\partial x_i \partial x_k} \right\} dt \\ &+ \sum_{j=1}^m \sum_{i=1}^d F_t^{i,j} \frac{\partial U}{\partial x_i} dW_t^j \end{aligned}$$

Example: Let X_t^1 and X_t^2 satisfy the scalar stochastic differentials

$$dX_t^i = e_t^i dt + f_t^i dW_t^i$$

for $i = 1, 2$ and let $U(t, x_1, x_2) = x_1 x_2$. Then the stochastic differential for the product process

$$Y_t = X_t^1 X_t^2$$

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depends on whether the Wiener processes W_t^1 and W_t^2 are independent or dependent. In the former case the differentials (1.1) can be written as the vector differential

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} e_t^1 \\ e_t^2 \end{pmatrix} dt + \begin{bmatrix} f_t^1 & 0 \\ 0 & f_t^2 \end{bmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}$$

and the transformed differential is

$$dY_t = \left(e_t^1 X_t^2 + e_t^2 X_t^1\right) dt + f_t^1 X_t^2 dW_t^1 + f_t^2 X_t^1 dW_t^2$$

In contrast, when two process driven by the same BM, i.e., $W_t^1 = W_t^2 = W_t$ the vector differential for (1.1) is

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} e_t^1 \\ e_t^2 \end{pmatrix} dt + \begin{pmatrix} f_t^1 \\ f_t^2 \end{pmatrix} dW_t$$

and there is an extra term $f_t^1 f_t^2 dt$ in the differential of Y_t, which is now

$$dY_t = \left(e_t^1 X_t^2 + e_t^2 X_t^1 + f_t^1 f_t^2\right) dt + \left(f_t^1 X_t^2 + f_t^2 X_t^1\right) dW_t \tag{1.2}$$

2 Linear SDEs

General form (scalar):

$$dX_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t, \tag{2.3}$$

with given coefficients, W_t and its associated σ-algebra A_t. Initial data X_{t_0} is A_{t_0} measurable.
Autonomous: if coefficients = consts against time.
Homogeneous: if $a_2 = b_2 = 0$:

$$dX_t = a_1(t)\, X_t dt + b_1(t)\, X_t\, dW_t, \tag{2.4}$$

solution with initial data $X_{t_0} = 1$, is called *fundamental solution*, Φ_{t,t_0}.
Q: what if $X_{t_0} = c$ where c is some non-zero constant?
Linear in narrow-sense: if $b_1 = 0$.

3 Narrow-sense linear SDE ($b_1 = 0$)

$$dX_t = (a_1(t)X_t + a_2(t))dt + (b_2(t))dW_t, \tag{3.5}$$

Fundamental solution ($a_2 = b_2 = 0$) is:

$$\Phi_{t,t_0} = \exp\{\int_{t_0}^{t} a_1(s)\, ds\}.$$

Using integrating factor idea, one wants to consider $d(\Phi_{t,t_0}^{-1} X_t)$, note

$$d\Phi_{t,t_0}^{-1} = d\exp\{-\int_{t_0}^{t} a_1(s)\,ds\} = -a_1(t)\Phi_{t,t_0}^{-1} dt.$$

now denoting $\Phi_{t,t_0} = \Phi$ for simplicity:

$$\begin{aligned} d(\Phi_{t,t_0}^{-1} X_t) &= [-a_1(t)(\Phi^{-1})X_t + (a_1 X_t + a_2)\Phi^{-1}]dt + b_2\Phi^{-1} dW_t, \\ &= a_2\Phi^{-1} dt + b_2\Phi^{-1} dW_t, \end{aligned} \tag{3.6}$$

integrating:

$$\Phi_{t,t_0}^{-1} X_t = X_{t_0} + \int_{t_0}^{t} a_2(s)\Phi_{s,t_0}^{-1} ds + \int_{t_0}^{t} b_2(s)\Phi_{s,t_0}^{-1} dW_s, \tag{3.7}$$

or:

$$X_t = \Phi_{t,t_0}[X_{t_0} + \int_{t_0}^{t} a_2(s)\Phi_{s,t_0}^{-1} ds + \int_{t_0}^{t} b_2(s)\Phi_{s,t_0}^{-1} dW_s]. \tag{3.8}$$

3.1 Eg1. Langevin equation and O-U

Langevin equation (a, b, constants):

$$dX_t = -aX_t dt + b dW_t,$$

solution:

$$X_t = e^{-at} X_0 + b\int_0^t e^{-a(t-s)} dW_s. \tag{3.9}$$

Lemma 3.1 *The process:*

$$V(t) = b\int_0^t e^{-a(t-s)} dW_s,$$

is Gaussian with covariance:

$$E[V(s)V(t)] = \rho(e^{-a|s-t|} - e^{-a|s+t|}), \quad \rho = b^2/(2a).$$

Sketch of proof: Consider $s, t \geq 0$. $V(t)$ is an approximation of sum $\sum f(t_j)(W_{j+1} - W_j)$, or sum of i.i.d. Gaussian r.v's, so it remains Gaussian. For a partition t_j's of $[0, t]$, write:

$$V(s) \approx b\sum_{[0,s]} e^{-a(s-t_k)}(W_{k+1} - W_k),$$

$$V(t) \approx b\sum_{[0,t]} e^{-a(t-t_k)}(W_{k+1} - W_k),$$

so:

$$E[V(s)V(t)] \approx b^2 \sum_{[0,\min(t,s)]} e^{-a(s+t)+2at_k}(t_{k+1} - t_k).$$

In the limit:

$$E[V(s)V(t)] = b^2 e^{-a(s+t)} \int_0^{\min(t,s)} e^{2a\tau} d\tau.$$

As $t \to \infty$, $E(V^2(t)) \to \rho$, limiting distribution $N(0,\rho)$. Process V is conditioned to zero at $t = 0$. To make it stationary, choose X_0 to be $N(0,\rho)$ independent of σ-algebra generated by $V(t)$, $t > 0$.

Lemma 3.2 *Langevin solution $X(t)$ in (3.9) with such X_0 gives O-U with covariance: $\rho e^{-a|t-s|}$.*

3.2 Eg2. Moments of SDE Solutions

We can also consider moments of SDE by Ito formula;
first moment $m(t) = E(X_t)$ from (2.3) directly:

$$m'(t) = a_1(t)m(t) + a_2(t). \tag{3.10}$$

Deriving another Ito SDE for X_t^2, where $dX = (a_1 X + a_2)dt + b_2 dW$ a then taking moment give equation for $P(t) = E(X_t^2)$:

$$P'(t) = 2a_1 P + 2m(t)a_2(t) + b_2^2(t). \tag{3.11}$$

Similarly higher moments. The solution is called "closed" at each level of moment.

4 General Linear SDE ($b_1 \neq 0$)

Using also integrating factor idea, only that fundamental solution of the homogeneous equation,

$$dX_t = a_1(t)X_t dt + b_1(t)X_t dW_t, \tag{4.12}$$

is stochastic.

Changing to Stratonovich form,

$$dX_t = (a_1 - \frac{1}{2}b_1^2)X_t dt + b_1 X_1 \circ dW_t,$$

we find:

$$\Phi_{t,t_0} = \exp\{\int_{t_0}^t [a_1(s) - \frac{1}{2}b_1^2(s)]ds + \int_{t_0}^t b_1(s)dW_s\}. \tag{4.13}$$

here we can remove $\circ$ as integrand is adapted (deterministic). Now by Ito formula,

$$d(\Phi_{t,t_0}^{-1}) = -(a_1(t) - \frac{1}{2}b_1^2(t))\Phi_{t,t_0}^{-1}dt - b_1\Phi_{t,t_0}^{-1}dW_t, \tag{4.14}$$

by (1.2)

$$
\begin{aligned}
d\left(\Phi_{t,t_0}^{-1}X_t\right) &= \left[\left(-a_1(t)+\frac{1}{2}b_1^2(t)\right)X_t\right]\Phi_{t,t_0}^{-1}dt \\
&+ ((a_1(t)X_t+a_2(t)) - \frac{1}{2}b_1(t)(b_1(t)X_t+b_2(t)))\Phi_{t,t_0}^{-1}dt \\
&+ \left[-b_1(t)\Phi_{t,t_0}^{-1}X_t + (b_1(t)X_t+b_2(t))\,\Phi_{t,t_0}^{-1}\right]dW_t \\
&= (a_2(t)-b_1(t)b_2(t))\,\Phi_{t,t_0}^{-1}dt + b_2(t)\Phi_{t,t_0}^{-1}dW_t
\end{aligned} \tag{4.15}
$$

integrating and taking Φ_{t,t_0}:

$$X_t = \Phi_{t,t_0}[X_{t_0} + \int_{t_0}^{t}(a_2-b_1b_2)\Phi_{t,t_0}^{-1}ds + \int_{t_0}^{t} b_2\Phi_{t,t_0}^{-1}dW_s]. \tag{4.16}$$

5 Reducible PDEs

For SDE,

$$dY_t = a(t,Y_t)dt + b(t,Y_t)dW_t, \tag{5.17}$$

We are looking for $X_t = U(t,Y_t)$, such that:

$$dX_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t. \tag{5.18}$$

Ito formula gives:

$$dU = (U_t + aU_y + \frac{1}{2}b^2U_{yy})dt + bU_ydW_t, \tag{5.19}$$

matching (5.18), (5.19):

$$(U_t + aU_y + \frac{1}{2}b^2U_{yy}) = a_1U + a_2, \tag{5.20}$$

$$bU_y = b_1U + b_2. \tag{5.21}$$

Two equations for U implies a compatibility condition on a and b.

Consider the Autonomous case:

$$dY_t = a(Y_t)dt + b(Y_t)dW_t, \tag{5.22}$$

and $X_t = U(Y_t)$. Eqns (5.20)-(5.21) reduce to (a_i, b_i consts in time):

$$a(y)U_y + \frac{1}{2}b^2(y)U_{yy} = a_1U(y) + a_2, \tag{5.23}$$

$$b(y)U_y = b_1U(y) + b_2. \tag{5.24}$$

If $b \neq 0$, $b_1 \neq 0$ (5.24) yields:

$$U(y) = Ce^{b_1B(y)} - b_2/b_1,\ B(y) = \int^y ds/b(s). \tag{5.25}$$

Plug (5.25) in (5.23):

$$(b_1 A(y) + \frac{1}{2} b_1^2 - a_1) C e^{b_1 B(y)} = a_2 - a_1 b_2 / b_1. \tag{5.26}$$

where

$$A(y) = a(y)/b(y) - b_y/2.$$

Diff. (5.26), multip. $b(y) e^{-b_1 B(y)} / b_1$,

$$b A_y + b_1 A + \frac{1}{2} b_1^2 - a_1 = 0 \tag{5.27}$$

diff. again:

$$b_1 A_y + (b A_y)_y = 0, \tag{5.28}$$

the compatibility condition on a and b. i.e. $(bA_y)_y$ is proportion to A_y.
To sum up:

$$U(y) = e^{b_1 B(y)}, \text{ if } b_1 \neq 0,$$

$$U(y) = b_2 B(y), \text{ if } b_1 = 0.$$

Then sub back to (5.23)-(5.24) to get other constant.

5.1 Example: Nonlinear SDE with local solution

Consider

$$dY_t = -\frac{1}{2} e^{-2Y_t} dt + e^{-Y_t} dW_t. \tag{5.29}$$

In this case, $A \equiv 0$, fully compatible for any b_1. Take $b_1 = 0$, $b_2 = 1$, $U = e^y$. Substituting this into (5.23) to find $a_1 = a_2 = 0$. Thus $X_t = e^{Y_t}$, and the resulting equation:

$$dX_t = dW_t,$$

solution:

$$X_t = W_t + e^{Y_0},$$

so:

$$Y_t = \ln(W_t + e^{Y_0}),$$

valid until time:

$$T = T(Y_0(\omega)) = \min\{t \geq 0 : W_t(\omega) + e^{Y_0(\omega)} = 0.\}$$

The example showed that nonlinear SDE solutions in general exist only for a finite time dependent on realizations. Like for determinsitic ODEs, we do not expect global existence of solutions without assumptions on the growth of nonlinearity in the equation.

5.2 Example: random logistic growth model:

$$dY_t = rY(t)(1 - Y(t))dt + Y(t)dW(t),$$

$r > 0$ constant growth rate, $Y(0) = Y_0$. Compatible if $b_1 = -1$, $b_2 = 0$, $a_1 = 1 - r$, $a_2 = r$. The transform is: $X = 1/Y$. X eqn:

$$dX(t) = ((1 - r)X(t) + r)dt - X(t)dW(t).$$

Solutions are:

$$Y(t) = \frac{\exp\{(r - 1/2)t + W(t)\}}{Y^{-1}(0) + r\int_0^t \exp\{(r - 1/2)t' + W(t')\}dt'}.$$

Solutions are global if $Y(0)r > 0$.

6 Project II (due 02/01/22)

II1. Let $X_t = \int_0^t f(s, \omega)\, dW_s$, show that e^{X_t} is a solution of SDE:

$$dY_t = \frac{1}{2} f^2(t, \omega)\, Y_t dt + f(t, \omega)\, Y_t\, dW_t,$$

and $e^{X_t - \frac{1}{2}\int_0^t f^2(s,\omega)\, ds}$ is a solution of SDE:

$$dY_t = f(t, \omega)\, Y_t\, dW_t.$$

II2. Derive the second moment equation for general linear Ito SDE, and find first and second moments of the Langevin equation.

II3. Generate the Ornstein-Uhlenbeck process numerically by discretizing the integral representation:

$$X_t = e^{-2t} X_0 + 2\int_0^t e^{-2(t-s)} dW_s,$$

with left hand rule (Ito) for a small grid size ds of your choice, for $t \in [0, 1]$. Here X_0 is $N(0, 1)$ r.v. independent of σ-algebra generated by $W(t)$, $t > 0$. Compute the covariance $E(X_t X_s)$ numerically and use that to help determine a choice of ds by comparing with exact covariance $e^{-2|t-s|}$. Plot a sample path of solution on $[0, 1]$.