Week 1

Statistics 251

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Where are we?

Course Overview

Why study probability?

Principles of counting

Problems and examples

Sample Spaces and Events

Relations and Operations

Venn diagrams and De Morgan's laws

Axioms of Probability

Welcome to Stat 251!

Instructors: Zhongjian Wang

- Prerequisites on website: email instructor with questions
- Course questions: Ed Discussion (link on Canvas)
- Weekly homeworks due Monday by 12:30pm
  - Submit on Gradescope (link on Canvas)
  - No late HW accepted, lowest HW dropped
- Midterm on April 24, Final scheduled by registrar
- Grade assigned based on maximum of
  - 20% HW + 40% Final + 40% Final
  - 20% HW + 30% Midterm + 50% Final

Course Outline

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1. Combinatorial analysis
2. Probability spaces and conditional probability
3. Discrete random variables
4. Continuous random variables
5. Change of variables
6. Limit theorems (Law of large numbers, Central limit theorem)

Where are we?

Disease prevalence

Suppose 50% of the population has a disease.

- In a group of 1 person, how many people should you expect have the disease?
- In a group of 100 people, how many people should you expect have the disease?
- In a group of 10000 people, how many people should you expect have the disease?

Suppose you test a group of N people and they all have the disease. How surprising is this for..

- $N = 1$?
- $N = 10$?
- $N = 100$?

Disease prevalence

Suppose 1% of the population has a disease.

- In a group of 1 person, how likely is it that someone has the disease?
- In a group of 100 people, how likely is it that someone has the disease?
- In a group of 10000 people, how likely is it that someone has the disease?

How do the answers change if 1% becomes 2%?

Where are we?

Addition and Multiplication Principles

Addition principle: Add up disjoint groups

- Q: If a class has 10 freshmen, 4 sophomores, 15 juniors, and 10 seniors, how many students does it have?

Multiplication principle: If independent choices are made, multiply the possible values of those choices.

- Q: If a class has 10 freshmen, 4 sophomores, 15 juniors, and 10 seniors, how many ways can we choose a project group with one member from each class?

Permutations

How many ways are there to order a deck of 52 cards?

If there are n people and n hats, how many ways are there to assign each person a hat?

Permutations

A **permutation** is a way to reorder a set of n distinct objects. The number of such permutations is

$$n! = n \cdot (n-1) \cdots 2 \cdot 1.$$

If there are $k < n$ people and n hats, how many ways are there to assign each person a hat?

Overcounting

How many ways are there to rearrange the letters in the word *PEAR*?

How many ways are there to rearrange the letters in the word *CHICAGO*?

Overcounting

Instead of counting the size of a set, it can be useful to **overcount by a fixed factor** and divide by that factor.

How many ways can a desk of indistinguishable 3 black, 4 red, and 5 blue cards be shuffled?

Combinations

How many length 4 strings of letters are there? What if the letters must be distinct?

How many sets of 4 distinct letters are there?

Combinations

The **binomial coefficient**

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

is the number of ways to choose a set of k items from n distinguishable items.

- $\frac{n!}{(n-k)!} = n \cdot (n-1) \cdots (n-k+1)$ is the number of ways to choose k **ordered** items
- This overcounts by $k!$, so we divide to get $\binom{n}{k}$.

Binomial Theorem

What is the coefficient in front of x^k in the expansion of $(x+1)^n$?

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The **binomial theorem** says that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + y^n.$$

Counting subsets

How many subsets does a set of size n have?

Where are we?

Counting poker hands

How many poker hands are there?

Counting poker hands

How many full house hands can one make in poker?

Counting poker hands

How many 2 pair hands can one make in poker?

Counting poker hands

How many hands have 4 cards of one suit and 1 card of another suit?

Counting numbers

How many 10 digit numbers have no consecutive digits the same?

Counting numbers

How many ways to assign a birthday to each of 23 distinct people?
What if no birthday can be repeated?

Counting problems

How many ways are there to divide a class of 60 students into 3 groups of 20?

Counting problems

How many ways can 10 identical beads be divided between 3 cups?

Where are we?

Sample Spaces

The set S of possible outcomes of an experiment is known as the **sample space** of the experiment.

- If the experiment consists of flipping a single coin, then

$$S = \{H, T\}.$$

- If the experiment consists of flipping two coins, then the sample space consists of the following four points:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

- If the experiment consists of measuring (in hours) the lifetime of a computer part, then S consists of all nonnegative real numbers; that is,

$$S = \{x : 0 \leq x \leq \infty\}$$

Events

Any subset E of the sample space is known as an **event**. In other words, an event is a subset of possible outcomes of the experiment.

When flipping a coin, $S = \{H, T\}$.

- If $E = \{H\}$, then E is the event that the coin lands heads up.
- If $F = \{T\}$, then F is the event that the coin lands tails up.

Events

When flipping two coins, $S = \{(H, H), (H, T), (T, H), (T, T)\}$:

- $E = (H, H), (H, T)$ is the event that the first coin is heads.

What is the event that there is at least one head?

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Axioms of Probability

Union

When flipping two coins

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

Let $E = \{(H, H), (H, T)\}$ be the event that the 1st coin is heads and $F = \{(T, H), (H, H)\}$ be the event that the 2nd coin is heads.

The **union** $E \cup F$ consists of all outcomes in either E or F. What is $E \cup F$ explicitly here?

Intersection

When flipping two coins

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

Let $E = \{(H, H), (H, T)\}$ be the event that the 1st coin is heads and $F = \{(T, H), (H, H)\}$ be the event that the 2nd coin is heads.

The **intersection** $E \cap F$ consists of all outcomes in both E and F. What is $E \cap F$ explicitly here?

Unions and intersections of more than two events

If E_1, E_2,... are events, then the union of these events, denoted by

$$\bigcup_{n=1}^{\infty} E_n$$

is the event consisting of all outcomes in **at least one** of the E_n.

The intersection of the events E_n, denoted by

$$\bigcap_{n=1}^{\infty} E_n,$$

consists of outcomes which are in **all** of the events E_n.

Mutually exclusive events

When flipping two coins, let E be the event the 1st coin is heads, and F the event the first coin is tails. What is $E \cap F$?

Mutually exclusive events

When flipping two coins, let E be the event the 1st coin is heads, and F the event the first coin is tails. What is $E \cap F$?

The event $E \cap F$ does not contain any outcomes and cannot occur. We call this the null event, denoted $\emptyset$.
If $E \cap F = \emptyset$, then E and F are said to be **mutually exclusive.**

Complement

When flipping two coins, let E be the event the 1st coin is heads, and F the event the first coin is tails. What is $E \cup F$?

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When flipping two coins, let E be the event the 1st coin is heads, and F the event the first coin is tails. What is $E \cup F$?

These events satisfy $E \cup F = S$ and $E \cap F = \emptyset$. In this case, we say that F is the **complement** of E, denoted $E = F^c$. Note $S^c = \emptyset$.

Where are we?

Venn diagrams

A Venn diagram is a graphical representation used to illustrate relations between events.

- Union, $E \cup F$
- Intersection, $E \cap F$
- Complement, E^c

De Morgan's Laws

The intersection and union operations satisfy some basic rules:

- Commutativity: $E \cup F = F \cup E$, $E \cap F = F \cap E$.
- Associativity: $(E \cup F) \cup G = E \cup (F \cup G)$, $(E \cap F) \cap G = E \cap (F \cap G)$.
- Distributive principle: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$, $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$.

DeMorgan's Laws

We can relate the complement operation to union and intersection:

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c.$$

Another analogue is:

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c.$$

Where are we?

What is probability? (frequentist version)

For a sample space S and an event $E \subset S$, define $\#(E, n)$ to be the number of times in the first n **repetitions** of the experiment that the event E occurs. Then the probability $\mathbb{P}(E)$ of the event E is defined as

$$\mathbb{P}(E) = \lim_{n \to \infty} \frac{\#(E, n)}{n}.$$

Note: We assume throughout this course that the probability of any event always exists. In other words, the limit always converges!

What is probability? (Bayesian version)

What if the experiment can only be run once?

For a sample space S and an event $E \subset S$, the probability $\mathbb{P}(E)$ is a number in $[0, 1]$ measuring the **subjective belief** or **reasonable expectation** of the chance that E will occur.

What is probability? (Bayesian version)

What if the experiment can only be run once?

For a sample space S and an event $E \subset S$, the probability $\mathbb{P}(E)$ is a number in $[0, 1]$ measuring the **subjective belief** or **reasonable expectation** of the chance that E will occur.

Sounds good, but what does that mean precisely...?

What is probability? (axiomatic version)

For a sample space S, a probability space on S assigns each event $E \subset S$ a number $\mathbb{P}(E)$ satisfying the following axioms:

1. $\mathbb{P}(E) \geq 0$
2. $\mathbb{P}(S) = 1$
3. For any sequence of mutually exclusive events E_1, E_2, ... (that is, events for which $E_i \cap E_j = \emptyset$ when $i \neq j$), we have

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

Probability spaces

Properties implied by the axioms:

1. $\mathbb{P}(\emptyset) = 0$
2. $\mathbb{P}(E) \leq 1$
3. If $E \subset F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$.
4. For any E, $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$.
5. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Example: Rolling a die

Example: measure how many times a die is rolled until you get a 6
$S = \{1, 2, 3, \ldots\}$

- Did we roll a 6 on our first try?
  $A = \{1\}$ (subset of $S = \{1, 2, 3, \ldots\}$)
- Did it take more than 5 rolls, to roll a 6?
  $B = \{6, 7, 8, \ldots\}$ (subset of $S = \{1, 2, 3, \ldots\}$)
- Did the first 6 occur on an even count roll?
  $C = \{2, 4, 6, \ldots\}$ (subset of $S = \{1, 2, 3, \ldots\}$)

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A probability space on S would specify the probability of each of these events, & any other event we might define.

For example, for a fair die, $\mathbb{P}(A) = \frac{1}{6}$, but $\mathbb{P}(A)$ could equal $\frac{1}{2}$ for an unfair die.