Lecture 9: Ito-Taylor Expansion II

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Abstract

General Form of Ito Taylor Expansion

0 Multi-indices

We shall call a row vector

$$\alpha = (j_1, j_2, \ldots, j_l)$$

where

$$j_i \in \{0, 1, \ldots, m\}$$

for $i \in \{1, 2, \ldots, l\}$ and $m = 1, 2, 3, \ldots$, a multi-index of length

$$l := l(\alpha) \in \{1, 2, \ldots\}$$

Here m will denote the number of components of the Wiener process under consideration. For completeness we denote by v the multi-index of length zero, that is with

$$l(v) := 0$$

Thus, for example,

$$l((1,0)) = 2 \text{ and } l((1,0,1)) = 3.$$

In addition, we shall write $n(\alpha)$ for the number of components of a multi-index which are equal to 0 . For example,

$$n((1,0,1)) = 1, \quad n((0,1,0)) = 2, \quad n((0,0)) = 2.$$

We denote the set of all multi-indices by $\mathcal{M}$, so

$$\mathcal{M} = \left\{ (j_1, j_2, \ldots, j_l) : j_i \in \{0, 1, \ldots, m\}, i \in \{1, \ldots, l\}, \text{ for } l = 1, 2, 3, \ldots \right\} \cup \{v\}. \quad (0.1)$$

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Given $\alpha \in \mathcal{M}$ with $l(\alpha) \geq 1$, we write $-\alpha$ and $\alpha-$ for the multi-index in $\mathcal{M}$ obtained by deleting the first and the last component, respectively, of α. Thus

$$-(1,0) = (0), (1,0)- = (1)$$
$$-(0,1,1) = (1,1), (0,1,1)- = (0,1).$$

Finally, for any two multi-indices $\alpha = (j_1, j_2, \ldots, j_k)$ and $\bar{\alpha} = (\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_l)$ we introduce an operation $*$ on $\mathcal{M}$ by

$$\alpha * \bar{\alpha} = (j_1, j_2, \ldots, j_k, \bar{j}_1, \bar{j}_2, \ldots, \bar{j}_l)$$

the multi-index formed by adjoining the two given multi-indices. We shall call this the concatenation operation. For example, for $\alpha = (0,1,2)$ and $\bar{\alpha} = (1,3)$ we have

$$\alpha * \bar{\alpha} = (0,1,2,1,3) \text{ and } \bar{\alpha} * \alpha = (1,3,0,1,2)$$

1 Hierarchical and Remainder sets

We call a subset $\mathcal{A} \subset \mathcal{M}$ an hierarchical set if $\mathcal{A}$ is nonempty;

$$\mathcal{A} \neq \emptyset$$

and the multi-indices in $\mathcal{A}$ are uniformly bounded in length:

$$\sup_{\alpha \in \mathcal{A}} l(\alpha) < \infty$$

and

$$-\alpha \in \mathcal{A} \quad \text{for each} \quad \alpha \in \mathcal{A} \backslash \{v\}$$

where v is the multi-index of length zero.

E.g.

$$\{v\}, \quad \{v, (0), (1)\}, \quad \{v, (0), (1), (1,1)\}$$

are hierarchical sets.

For any given hierarchical set $\mathcal{A}$ we define the remainder set $\mathcal{B}(\mathcal{A})$ of $\mathcal{A}$ by

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M} \backslash \mathcal{A} : -\alpha \in \mathcal{A}\}$$

E.g., for SDE driven by 1D Wiener process,

$$\mathcal{B}(\{v\}) = \{(0), (1)\}, \quad \mathcal{B}(\{v, (0), (1)\}) = \{(0,0), (0,1), (1,0), (1,1)\}$$

and

$$\mathcal{B}(\{v, (0), (1), (1,1)\}) = \{(0,0), (0,1), (1,0), (0,1,1), (1,1,1)\}$$

Given r is positive integer,

$$\Gamma_r = \{\alpha \in \mathcal{M} : l(\alpha) \le r\}, \tag{1.2}$$

and

$$\Lambda_r = \{\alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \le r\} \tag{1.3}$$

are hierarchical set.

2 Ito-Taylor Expansion

2.1 Recall

$$\begin{aligned} X_t &= X_0 + a(X_0)\int_0^t ds + b(X_0)\int_0^t dW_s \\ &\quad + (L^1 b)(X_0)\int_0^t \int_0^s dW_z dW_s + R_1, \end{aligned} \tag{2.4}$$

remainder:

$$\begin{aligned} R_1 &= \int_0^t \int_0^s (L^0 a)(X_z) dz ds + \int_0^t \int_0^s (L^1 a)(X_z) dW_z ds \\ &\quad + \int_0^t \int_0^s (L^0 b)(X_z) dz dW_s + \int_0^t \int_0^s \int_0^z (L^0 L^1 b)(X_u) du dW_z dW_s \\ &\quad + \int_0^t \int_0^s \int_0^z (L^1 L^1 b)(X_u) dW_u dW_z dW_s, \end{aligned} \tag{2.5}$$

where

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2}\sum_{k,l=1}^{d}\sum_{j=1}^{m} b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^l}$$

and

$$L^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k}.$$

If applied to a function f,

$$\begin{aligned} f(X_t) &= f(X_0) + (a(X_0) f'(X_0) + \frac{1}{2} b^2(X_0) f''(X_0)) \int_0^t ds + b(X_0) f'(X_0) \int_0^t dW_s \\ &\quad + (L^1(bf'))(X_0) \int_0^t \int_0^s dW_z dW_s + R_1, \end{aligned} \tag{2.6}$$

remainder:

$$
\begin{aligned}
R_1 \;=\; & \int_0^t \int_0^s (L^0(af' + \frac{1}{2}b^2 f''))(X_z) dz ds + \int_0^t \int_0^s (L^1(af' + \frac{1}{2}b^2 f''))(X_z) dW_z ds \\
& + \int_0^t \int_0^s (L^0(bf'))(X_z) dz dW_s + \int_0^t \int_0^s \int_0^z (L^0 L^1(bf'))(X_u) du dW_z dW_s \\
& + \int_0^t \int_0^s \int_0^z (L^1 L^1(bf'))(X_u) dW_u dW_z dW_s. \qquad (2.7)
\end{aligned}
$$

2.2 Statement

Let ρ and τ be two stopping times with

$$t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T$$

w.p.1; let $\mathcal{A} \subset \mathcal{M}$ be an hierarchical set; and let $f : \Re^+ \times \Re^d \to \Re$. Then the Ito-Taylor expansion

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}} I_\alpha \left[f_\alpha(\rho, X_\rho) \right]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha \left[f_\alpha(\cdot, X.) \right]_{\rho,\tau} \qquad (2.8)$$

holds, provided all of the derivatives of f, a and b and all of the multiple Ito integrals appearing in (2.8) exist.

2.3 Examples

1. We take the hierarchical set $\mathcal{A} = \{v\}$, which has the remainder set

$$\mathcal{B}(\{v\}) = \{(0), (1), \ldots, (m)\}$$

Then,

$$
\begin{aligned}
f(\tau, X_\tau) &= I_v \left[f_v(\rho, X_\rho) \right]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\{v\})} I_\alpha \left[f_\alpha(\cdot, X.) \right]_{\rho,\tau} \\
&= f(\rho, X_\rho) + \int_\rho^\tau L^0 f(s, X_s)\, ds + \sum_{j=1}^m \int_\rho^\tau L^j f(s, X_s)\, dW_s^j.
\end{aligned}
$$

This is the Ito Formula.

2. (2.6) with remainder term (2.7) can be from (2.8) with hierarchical set $\mathcal{A} = \Lambda_2 = \{v, (0), (1), (1,1)\}$.

2.4 Proof (not a stopping time version)

Lemma Let $\alpha, \beta \in \mathcal{M}$. Then

$$I_\alpha \left[f_\beta(\cdot, X.)\right]_{\rho,\tau} = I_\alpha \left[f_\beta\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha}\left[f_{(j)*\beta}(\cdot, X.)\right]_{\rho,\tau} \tag{2.9}$$

Proof:

First, Ito formula yields,

$$f\left(\tau, X_\tau\right) = f\left(\rho, X_\rho\right) + \sum_{j=0}^{m} I_{(j)}\left[L^j f(\cdot, X.)\right]_{\rho,\tau}.$$

For $l(\alpha) = 0$ we have $\alpha = v$. Hence,

$$\begin{aligned} I_\alpha \left[f_\beta(\cdot, X.)\right]_{\rho,\tau} &= f_\beta\left(\tau, X_\tau\right) \\ &= f_\beta\left(\rho, X_\rho\right) + \sum_{j=0}^{m} I_{(j)}\left[L^j f_\beta(\cdot, X.)\right]_{\rho,\tau} \\ &= I_\alpha \left[f_\beta\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha}\left[f_{(j)*\beta}(\cdot, X.)\right]_{\rho,\tau} \end{aligned}$$

Now let $l(\alpha) = k \geq 1$, where $\alpha = (j_1, \ldots, j_k)$. Then,

$$\begin{aligned} I_\alpha \left[f_\beta(\cdot, X.)\right]_{\rho,\tau} =& I_{(j_k)}\left[I_{\alpha-}\left[\left(f_\beta(\cdot, X.)\right]_{\rho,\cdot}\right]_{\rho,\tau} \\ =& I_{(j_k)}\left[I_{\alpha-}\left[\left(f_\beta\left(\rho, X_\rho\right)\right]_{\rho,\cdot}\right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j_k)}\left[I_{(j)*\alpha-}\left[f_{(j)*\beta}(\cdot, X.)\right]_{\rho,}\right]_{\rho,\tau} \\ =& I_\alpha \left[f_\beta\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha}\left[f_{(j)*\beta}(\cdot, X.)\right]_{\rho,\tau}. \end{aligned}$$

Main Theorem We shall prove by induction on,

$$l_1(\mathcal{A}) = \sup_{\alpha\in\mathcal{A}} l(\alpha).$$

For $l_1(\mathcal{A}) = 0$ we have $\mathcal{A} = \{v\}$ with the remainder set

$$\mathcal{B}(\mathcal{A}) = \{(0), (1), \cdots, (m)\}$$

Then

$$f\left(\tau, X_\tau\right) = \sum_{\alpha\in\mathcal{A}} I_\alpha \left[f_\alpha\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{\alpha\in\mathcal{B}(\mathcal{A})} I_\alpha \left[f_\alpha(\cdot, X.)\right]_{\rho,\tau}$$

Now let $l_1(\mathcal{A}) = k \geq 1$. If we set

$$\mathcal{E} = \{\alpha \in \mathcal{A} : l(\alpha) \leq k-1\}$$

which is an hierarchical set, then by the inductive assumption we obtain

$$f(\tau, X_\tau) = \sum_{\alpha\in\mathcal{E}} I_\alpha \left[f_\alpha(\rho, X_\rho)\right]_{\rho,\tau} + \sum_{\alpha\in\mathcal{B}(\mathcal{E})} I_\alpha \left[f_\alpha(\cdot, X.)\right]_{\rho,\tau}$$

Since $\mathcal{A}$ is an hierarchical set with $l_1(\alpha) = k$,

$$\mathcal{A}\backslash\mathcal{E} \subseteq \mathcal{B}(\mathcal{E})$$

For $\beta = \alpha \in \mathcal{A}\backslash\mathcal{E}$ so we can rewrite as

$$\begin{aligned}
f(\tau, X_\tau) =& \sum_{\alpha\in\mathcal{E}} I_\alpha \left[f_\alpha(\rho, X_\rho)\right]_{\rho,\tau} + \sum_{\alpha\in\mathcal{A}|\mathcal{E}} I_\alpha \left[f_\alpha(\cdot, X.)\right]_{\rho,\tau} \\
&+ \sum_{\alpha\in\mathcal{B}(\mathcal{E})\backslash(\mathcal{A}\backslash\mathcal{E})} I_\alpha \left[f_\alpha(\cdot, X.)\right]_{\rho,\tau} \\
=& \sum_{\alpha\in\mathcal{E}} I_\alpha \left[f_\alpha(\rho, X_\rho)\right]_{\rho,r} \\
&+ \sum_{\alpha\in\mathcal{A}\backslash\mathcal{E}} \left[I_\alpha \left[f_\alpha(\rho, X_\rho)\right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha} \left[f_{(j)*\alpha}(\cdot, X.)\right]_{\rho,\tau} \right] \\
&+ \sum_{\alpha\in\mathcal{B}(\mathcal{E})\backslash(\mathcal{A}|\varepsilon)} I_\alpha \left[f_\alpha(\cdot, X.)\right]_{\rho,\tau} \\
=& \sum_{\alpha\in\mathcal{A}} I_\alpha \left[f_\alpha(\rho, X_\rho)\right]_{\rho,\tau} + \sum_{\alpha\in\mathcal{B}_1} I_\alpha \left[f_\alpha(\cdot, X.)\right]_{\rho,\tau}
\end{aligned}$$

Now note,

$$\begin{aligned}
\mathcal{B}_1 =& [\mathcal{B}(\mathcal{E})\backslash(\mathcal{A}\backslash\mathcal{E})] \bigcup \left[\bigcup_{j=0}^{m} \{(j) * \alpha \in \mathcal{M} : \alpha \in \mathcal{A}\backslash\mathcal{E}\} \right] \\
=& [\{\alpha \in \mathcal{M}\backslash\mathcal{E} : -\alpha \in \mathcal{E}\} \backslash \{\alpha \in \mathcal{M}\backslash\mathcal{E} : \alpha \in \mathcal{A}\}] \\
& \bigcup \{\alpha \in \mathcal{M} : -\alpha \in \mathcal{A}\backslash\mathcal{E}\} \\
=& \{\alpha \in \mathcal{M}\backslash\mathcal{A} : -\alpha \in \mathcal{E}\} \bigcup \{\alpha \in \mathcal{M}\backslash\mathcal{A} : -\alpha \in \mathcal{A}\backslash\mathcal{E}\} \\
=& \{\alpha \in \mathcal{M}\backslash\mathcal{A} : -\alpha \in \mathcal{A}\} \\
=& \mathcal{B}(\mathcal{A})
\end{aligned}$$