Math 18500 Week 5: Second Order Inhomogeneous Equations

Exponential Response. We now turn our attention to inhomogeneous second order equations

$$my'' + ly' + ky = f(t)$$

where m, l, and k are constants, and $f(t)$ is a function of t. Recall that equations like this can be written in operator form as

$$\mathcal{O}y = f,$$

where

$$\mathcal{O} = mD^2 + lD + k$$

is a linear differential operator.

Before we get started solving inhomogeneous equations, recall the three crucial properties of linear equations:

(1) Suppose that $y_1(t)$ and $y_2(t)$ are two solutions of a linear homogeneous equation:

$$\mathcal{O}y_1 = 0$$
$$\mathcal{O}y_2 = 0.$$

If c_1 and c_2 are arbitrary constants, and

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

then $y(t)$ is a solution of the same homogeneous equation:

$$\mathcal{O}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{O}y_1 + c_2 \mathcal{O}y_2 = 0 + 0 = 0$$

In other words, *solutions of homogeneous equations can be superimposed.*

(2) Suppose that $y_1(t)$ and $y_2(t)$ are solutions of linear inhomogeneous equations with different right hand sides:

$$\mathcal{O}y_1 = f_1$$
$$\mathcal{O}y_2 = f_2$$

If c_1 and c_2 are arbitrary constants, and

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

then $y(t)$ is a solution of the inhomogeneous equation

$$\mathcal{O}\left[c_1 y_1 + c_2 y_2\right] = c_1 f_1 + c_2 f_2$$

where the right hand side is a linear combination of the right hand sides of the original equations.

(3) Suppose that $y_p(t)$ is a *particular* solution of an inhomogeneous equation,

$$\mathcal{O}y_p = f$$

If $y_h(t)$ is any solution of the corresponding *homogeneous* equation,

$$\mathcal{O}y_h = 0.$$

and if

$$y(t) = y_p(t) + y_h(t),$$

then $y(t)$ is a solution of the original inhomogeneous equation:

$$\mathcal{O}y = \mathcal{O}y_p + \mathcal{O}y_h = f + 0 = f$$

From property (3) you can see that the only difficult part of solving an inhomogeneous equation is finding *one particular solution.* Once a particular solution y_p has been found, more general solutions can always be constructed by adding solutions of the homogeneous equation.

As a simple example to get us started, suppose we want to solve an inhomogeneous equation like

$$y'' + 4y' + 3y = e^t.$$

In general, we can often find particular solutions of equations of the form

$$my'' + ly' + ky = e^{rt},$$

using an ansatz of the form

$$y = Ce^{rt},$$

where C is an unspecified constant (or *undetermined coefficient*).

In the case of the equation

$$y'' + 4y' + 3y = e^t$$

we can try the ansatz $y = Ce^t$. This gives us

$$Ce^t + 4Ce^t + 3Ce^t = e^t.$$

Dividing by e^t we find that

$$8C = 1,$$

so $C = \frac{1}{8}$. This gives us a particular solution,

$$y_p = \frac{1}{8}e^t.$$

Once we have a particular solution, more general solutions can be obtained by adding solutions of the corresponding homogeneous equation

$$y'' + 4y' + 3y = 0$$

The general solution of this equation works out to be

$$y_h = c_1 e^{-t} + c_2 e^{-3t},$$

and we can conclude that

$$y = y_p + y_h = \frac{1}{8}e^t + c_1 e^{-t} + c_2 e^{-3t}$$

is a solution of the original inhomogeneous equation. [1]

This idea allows us to solve arbitrary inhomogeneous initial value problems (once a particular solution is known!). For example, to solve the initial value problem

$$y'' + 4y' + 3y = e^t \ , \ \ y(0) = 0 \ , \ \ y'(0) = 0$$

we can just write out the solution we found above and its derivative,

$$y = \frac{1}{8}e^t + c_1 e^{-t} + c_2 e^{-3t}$$

$$y' = \frac{1}{8}e^t - c_1 e^{-t} - 3c_2 e^{-3t}$$

Setting $t = 0$ in both equations gives us a system of equations for c_1 and c_2,

$$0 = y(0) = \frac{1}{8} + c_1 + c_2$$

$$0 = y'(0) = \frac{1}{8} - c_1 - 3c_2$$

Adding the equations, we find that

$$0 = \frac{1}{4} - 2c_2 \implies c_2 = \frac{1}{8}$$

and substituting in the first equation, we find that

$$0 = \frac{1}{8} + c_1 + \frac{1}{8} \implies c_1 = -\frac{1}{4}$$

So, the solution of the initial value problem is

$$y = \frac{1}{8}e^t - \frac{1}{4}e^{-t} + \frac{1}{8}e^{-3t}.$$

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[1] In fact, this is the *general* solution of the inhomogeneous equation! If y was any other solution, then we would have

$$\mathcal{O}y = \mathcal{O}y_p = f$$

$$\mathcal{O}(y - y_p) = \mathcal{O}y_1 - \mathcal{O}y_2 = f - f = 0.$$

Therefore, $y_h = y - y_p$ is a solution of the homogeneous equation, and

$$y = y_p + y_h$$

To reiterate, the only difficult part of solving inhomogeneous equations is finding particular solutions!

As a next simplest case, suppose we are given an inhomogeneous equation

$$my'' + ly' + ky = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where the right hand side is a *linear combination of exponentials*. Equations of this form can be solved by applying crucial property (2) of linear equations.

For example, consider the equation

$$y'' + 4y' + 3y = 5e^t - 7e^{3t}.$$

To find a particular solution of this equation, we first seek solutions $y_1(t)$ and $y_2(t)$ of the equations

$$y_1'' + 4y_1' + 3y_1 = e^t$$
$$y_2'' + 4y_2' + 3y_2 = e^{3t}.$$

If we can find theses solutions, then the linear combination $y(t) = 5y_1(t) - 7y_2(t)$ will be a solution of the equation

$$y'' + 4y' + 3y = 5e^t - 7e^{3t}.$$

Of course, we have already seen that $y_1 = \frac{1}{8}e^t$ is a solution of the first equation. To find a solution of the second equation, we can try an ansatz of the form $y_2 = Be^{3t}$, where B is an undetermined constant. This leads to the equation

$$9Be^{3t} + 12Be^{3t} + 3Be^{3t} = e^{3t},$$

and solving this equation gives the value $B = \frac{1}{24}$, so

$$y_2 = \frac{1}{24}e^{3t}.$$

Therefore,

$$y = 5y_1(t) - 7y_2(t) = \frac{5}{8}e^t - \frac{7}{24}e^{3t}$$

is the desired particular solution.

Sinusoidal Response. The methods of the previous seciton can also be applied when the right hand side is a *sinusoid*, because sinusoids are linear combinations of *complex exponentials*.

For example, to find a particular solution of the equation

$$y'' + y' + 3y = \cos t$$

we can use Euler's formula to rewrite the right hand side of the equation as a sum of complex exponentials:

$$\cos(t) = \frac{1}{2}(\cos t + i \sin t) - \frac{1}{2}(\cos t - i \sin t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}.$$

This transforms the right hand side of our equation into a linear combination of complex exponentials,

$$y'' + y' + 3y = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it},$$

and we can solve this equation using the same procedure as before, by solving the equations

$$z'' + z' + 3z = e^{it}$$
$$w'' + w' + 3w = e^{-it}$$

individually and then superimposing the solutions:

$$y_p = \frac{1}{2}z(t) + \frac{1}{2}w(t)$$

However, it is not really necessary to solve both of these equations. Instead, we can just solve the *single* complex equation

$$z'' + z' + 3z = e^{it}$$

and write the solution as

$$z(t) = y_1(t) + iy_2(t).$$

Then we will have

$$y_1'' + iy_2'' + y_1' + iy_2' + 3(y_1 + iy_2) = \cos t + i \sin t.$$

Separating this equation into its real and imaginary parts, we find that

$$y_1'' + y_1' + 3y_1 = \cos t$$
$$y_2'' + y_2' + 3y_2 = \sin t$$

Therefore, the real part of the complex solution,

$$y_p = y_1(t) = \mathrm{Re}\,[z(t)]$$

will be a particular solution of the equation we were trying to solve.

Even though this strategy works a bit more nicely than the first idea (superimposing two complex solutions), it is worth observing how the two methods are related. The key observation here is that we can find a particular solution of the equation

$$w'' + w' + 3w = e^{-it}$$

by taking the *complex conjugate* of a solution of the equation

$$z'' + z' + 3z = e^{it}$$

In other words,

$$w(t) = \overline{z(t)},$$

so when we superimpose the solutions we get the particular solution

$$y_p = \frac{1}{2}z(t) + \frac{1}{2}w(t) = \frac{1}{2}\left(z(t) + \overline{z(t)}\right) = \mathrm{Re}\,[z(t)]$$

Method of Undetermined Coefficients. So far we have always been able to solve equations of the form

$$my'' + by' + ky = e^{rt}$$

using ansatz of the form

$$y_p = Ae^{rt}$$

However, there are some exceptional cases where this method will fail.

For example, suppose we want to solve an inhomogeneous equation like

$$y'' + 4y' + 3y = e^{-3t},$$

where the exponential on the right hand side is a solution of the corresponding homogeneous equation,

$$y'' + 4y' + 3y = 0.$$

In this case, when we use the standard ansatz $y = Ae^{-3t}$, we get

$$9Ae^{-3t} - 12Ae^{-3t} + 3Ae^{-3t} = e^{-3t},$$

which leads to the inconsistent equation

$$0 = e^{-3t}.$$

So, our method fails to produce a particular solution in this case.

In cases like this, one strategy is to resort to repeated integration. Writing the equation in operator form

$$(D^2 + 4D + 3)y = e^{-3t}$$

and factoring the operator, we obtain

$$(D + 3)(D + 1)y = e^{-3t}$$

We then set

$$(D + 1)y = u$$

and solve the equation

$$(D + 3)u = u' + 3u = e^{-3t}.$$

Using the integrating factor $J = e^t$ we obtain

$$(e^{3t}u)' = 1$$
$$e^{3t}u = t + C_0$$
$$u = te^{-3t} + C_0e^{-3t}$$

This leaves us with the equation

$$(D + 1)y = u = te^{-3t}$$

or

$$y' + y = te^{-3t} + C_0e^{-3t}.$$

Using the integrating factor $J = e^t$ we obtain

$$(e^t y)' = te^{-2t} + C_0e^{-2t}$$

$$e^t y = \int te^{-2t} + \int C_0e^{-2t} = -\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t} + \int C_0e^{-2t} = -\frac{1}{2}te^{-2t} + C_1e^{-2t} + C_2$$

where C_1 and C_2 are arbitrary constants. This gives the general solution, This gives the *general* solution,

$$y = -\frac{1}{2}te^{-3t} + C_1e^{-3t} + C_2e^{-t}$$

Obviously, this is not a method you would want to apply routinely!

A more clever way to solve the equation is to notice that if we apply an operator

$$mD^2 + lD + k$$

to a function of the form

$$y = p(t)e^{rt},$$

where

$$p(t) = p_0 + p_1t + p_2t^2 + \cdots + p_nt^n$$

is a polynomial function of t, then we obtain

$$(mD^2 + lD + k)p(t)e^{rt} = q(t)e^{rt}$$

where

$$q(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_n t^n$$

is *also* a polynomial function of t.

This observation suggests that it might always possible to solve equations of the form

$$my'' + ly' + ky = q(t)e^{rt} = (q_0 + q_1 t + q_2 t^2 + \cdots + q_n t^n)e^{rt}$$

using an ansatz

$$y = p(t)e^{rt} = (p_0 + p_1 t + p_2 t^2 + \cdots + p_d t^d)e^{rt}$$

Making a guess of this form is called the **method of undetermined coefficients**.

Notice that the degree d of $p(t)$ sometimes needs to be *larger* than n. We have already seen this in the example

$$y'' + 4y' + 3y = e^{-3t}.$$

where $n = 0$ and a polynomial of degree $d = 1$ was required. It turns out that in general we must take the degree of $p(t)$ to be $d = n + 1$ if r is a root of the auxiliary equation

$$m\lambda^2 + l\lambda + k = 0$$

and $d = n + 2$ if it is a *repeated* root.

For example, consider the equation

$$y'' - 2y' + y = e^t$$

In this case, 1 is a repeated root of the auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0,$$

so we must try an ansatz of the form

$$y = (At^2 + Bt + C)e^t$$

where A, B, and C are unknown constants.

This seems like it might be a lot of work! However, notice that the functions e^t and te^t are solutions of the corresponding homogeneous equation

$$y_h'' + 2y_h' + y_h = 0.$$

Since we are only interested in finding *one* particular solution of

$$y'' - 2y' + y = e^t,$$

we can subtract off these homogeneous solutions and try the *simpler* ansatz

$$y = At^2 e^t.$$

Evaluating the derivatives,

$$y' = 2Ate^t + At^2 e^t$$
$$y'' = 2Ae^t + 4Ate^t + At^2 e^t$$

and substituting, we find that

$$y'' - 2y' + y = At^2 e^t + 4Ate^t + 2Ae^t - 4Ate^t - 2At^2 e^t + At^2 e^t = 2Ae^t$$

and therefore we can obtain a particular solution of the equation

$$y'' - 2y' + y = e^t,$$

by taking

$$A = \frac{1}{2}$$

or

$$y = \frac{1}{2}t^2 e^t.$$

Forced Oscillations. Consider a damped oscillator which is governed by an equation

$$my'' + ly' + ky = f(t).$$

Our goal in this lecture is to understand in detail what happens when the forcing term is a *sinusoid*,

$$f(t) = A_d \cos(\omega_d t).$$

with some amplitude A_d (the *driving amplitude*) and frequency ω_d (the *driving frequency*).

We have already seen that the equation

$$my'' + ly' + ky = \cos(\omega_d t)$$

can be solved by recognizing it as the real part of a *complex* equation

$$mz'' + lz' + kz = e^{i\omega_d t}.$$

A particular solution of this inhomogeneous equation can be found as a complex exponential of the form

$$z(t) = Ze^{i\omega t},$$

where Z is a complex number. To solve for Z, we substitute, obtaining an algebraic equation

$$mZ(i\omega)^2 + lZ(i\omega) + kZ = 1,$$

whose solution is

$$Z = Z(\omega) = \frac{1}{(k - m\omega^2) + i(l\omega)}$$

The complex-valued function $Z(\omega)$ is called the *frequency response function* of the oscillator. It is useful to write this function in polar form, as

$$Z = Ge^{-i\phi}.$$

The real quantities $G = G(\omega)$ and $\phi = \phi(\omega)$ are referred to as the *amplitude gain* and *phase shift*, respectively.

With this notation, we have

$$\mathrm{Re}\,[z(t)] = \mathrm{Re}\left[Ge^{-i\phi}e^{i\omega t}\right] = \mathrm{Re}\left[Ge^{i(\omega t - \phi)}\right] = G\cos(\omega t - \phi)$$

Therefore, the general solution takes the form

$$y(t) = y_p(t) + y_h(t) = G\cos(\omega t - \phi) + y_h(t),$$

where $y_h(t)$ is a solution of the homogeneous equation. For a damped oscillator, any solution of the homogeneous equation dies off at an exponential rate. Because of this, the term $y_h(t)$ is called a *transient*, and the term $y_p(t)$ is called the *steady state* response.

More generally, the general solution of

$$my'' + ly' + ky = A_d \cos(\omega_d t).$$

will be

$$y = A\cos(\omega t - \phi) + y_h(t)$$

where the steady state response has amplitude

$$A = GA_d.$$

In other words, the amplitude gain is the ratio of the response amplitude to the driving amplitude:

$$G = \frac{A}{A_d}.$$

This explains why we call G the amplitude *gain*.

To summarize: the response of a damped oscillator to a sinusoidal input is the sum of a sinusoidal steady state response and a transient. The sinusoidal part of the responce has the *same frequency* as the input, but its amplitude may be larger or smaller, and it may be phase-shifted (its peaks may not line up with the peaks of the input).

As a function of ω, the amplitude gain is given by the explicit formula

$$G(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + (l\omega)^2}}.$$

Notice that as ω tends to ∞, the amplitude gain tends to 0. This is familiar to any child who has been on a swing - if you sit and swing your legs rapidly back and forth, barely anything happens. To get the swing moving, you have to instead match the frequency of your leg motion to the *resonant frequency* of the swing.

By definition, the resonant frequency of an oscillator is the frequency at which the *maximum* amplitude gain is realized. We can calculate the resonant frequency of any oscillator theoretically, by using calculus to find the *maximum value* of the gain function $G(\omega)$, or equivalently the minimum value of

$$\frac{1}{G(\omega)^2} = (k - m\omega^2)^2 + (l\omega)^2 = m^2\omega^4 + (l^2 - 2mk)\omega^2 + k^2.$$

We do this by taking the first derivative with respect to ω:

$$\frac{d}{d\omega}\frac{1}{G(\omega)^2} = 4m^2\omega^3 + 2(l^2 - 2mk)\omega$$

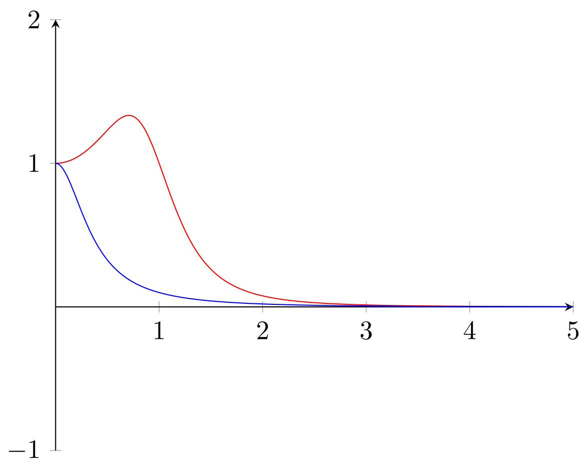
Setting this equal to zero, we see that (for $\omega \geq 0$) there are two critical points,

$$\omega = 0 \text{ and } \omega = \sqrt{\frac{2km - l^2}{2m^2}}.$$

The second critical point only exists when the quantity inside the square root is real:

$$l^2 \leq 2km.$$

If $l^2 \geq 2km$, then $\omega = 0$ is a global maximum - otherwise it is a local minimum. Here are plots of $G(\omega)^2$ which illustrate these two cases:



Recall that critical damping occurs when $l^2 = 4mk$. It follows that overdamped oscillators (and some underdamped oscillators) do not resonate - resonance only occurs if the oscillator is *highly underdamped*.

So far we have only discussed the case of a *damped* oscillator ($b > 0$). In the case of an undamped *harmonic* oscillator, the phenomenon of resonance presents in a slightly different way. In this case, the amplitude gain is given by

$$G = \frac{1}{k - m\omega^2}.$$

In this case, the amplitude gain becomes *infinite* when

$$\omega = \sqrt{\frac{k}{m}} = \omega_0,$$

i.e. when the input frequency matches the natural frequency of the oscillator.

In particular, the usual exponential ansatz will *fail* in the resonant case $\omega = \omega_0$. To illustrate this, consider the simplest case

$$y'' + y = \cos t.$$

In this case, the complexified equation is

$$z'' + z = e^{it},$$

and when we try the ansatz

$$z(t) = Ze^{it}$$

we get the inconsistent equation

$$-Ze^{it} + Ze^{it} = 0 = e^{it}.$$

To get around the difficulty, we can use the ansatz specified by the method of undetermined coefficients,

$$z = Ate^{it}.$$

Differentiating this ansatz we obtain

$$z' = Ae^{it} + Aite^{it}$$
$$z'' = 2Aie^{it} - Ate^{it}$$

This leads to the equation

$$2Aie^{it} - Ate^{it} + Ate^{it} = e^{it}$$

and we conclude that

$$A = \frac{1}{2i}$$

Therefore a particular solution of the complexified equation is

$$z = \frac{t}{2i}e^{it} = \frac{1}{2}t\sin t - \frac{it}{2}\cos t.$$

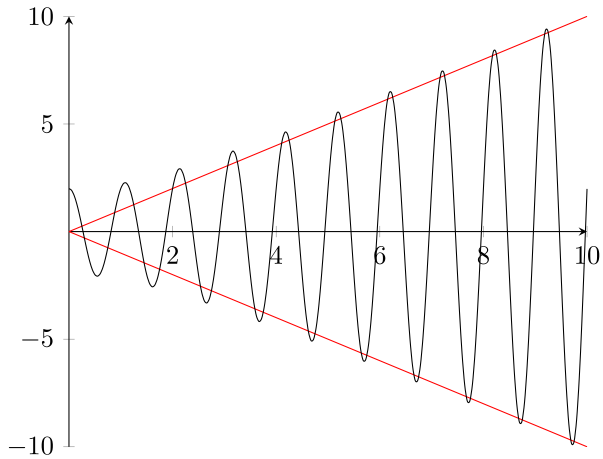
Taking the real part, we obtain the particular solution

$$y_p = \frac{1}{2}t\sin t,$$

and conclude that the general solution is

$$y = y_p + y_h = \frac{1}{2}t\sin t + a\cos t + b\sin t.$$

Notice that any such solution oscillates between larger and larger values in the limit as $t \to \infty$:



In this sense, harmonic oscillators are *unstable* -z a small (bounded) input can potentially lead to an arbitrarily large (unbounded) output. From a practical point of view, this instability also manifests itself for oscillators which are insufficiently damped - such an oscillator can be forced to oscillate very violently, even by an input of modest amplitude, if the input frequency is close to the resonant frequency. This can have catastrophic results (bridges collapsing, overloaded circuits, etc.) and responsible engineers must take it into account.

For some interesting examples of resonance, see the following videos:

https://www.youtube.com/watch?v=4pEVl2Q86QM&ab_channel=xmdemo
https://www.youtube.com/watch?v=3mclp9QmCGs&ab_channel=SimonLesp%C3%A9rance