Lecture 19: Expectations of sums

Statistics 251

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Lecture Outline

Revision and derivation

Sample Mean

Examples

Where are we?

Revision: Expected Value of a function of a r.v.

If X and Y have a joint probability mass function $p(x, y)$, then

$$E[g(X, Y)] =$$

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What about $E\left[X_1 + \cdots + X_n\right]$?

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Note: when the distribution mean μ is unknown, the sample mean is often used in statistics to estimate it.

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Examples

Example: hat matches

Suppose that N people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects one. Find the expected number of people that select their own hat.

Example: A Summation Formula

Consider any nonnegative, integer-valued random variable X. If, for each $i \geq 1$, we define

$$X_i = \begin{cases} 1 & \text{if } X \geq i \\ 0 & \text{if } X < i \end{cases}$$

then $E[X] = \sum_{i=1}^{\infty} E\left(X_i\right)$

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a useful identity.

Example: Sorting elements

Suppose that n elements, $1, 2, \cdots, n$ must be stored in a computer in the form of an ordered list. Each unit of time, a request will be made for one of these elements i being requested, independently of the past, with known probability $P(i), i \geq 1$, $\sum_i P(i) = 1$. What ordering minimizes the average position in the line of the element requested?

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$$P_O\{X \geq k\} = \sum_{j=k}^{n} P\left(i_j\right)$$