Lecture 3: Integral

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Abstract

Ito Integral, Ito Formula, Stratonovich Integral

1 Motivation and Definition

1.1 SDE of OU

Recall in last lecture, for OU process with coefficient γ, $\forall \tau \geq 0$, $X(t+\tau) - e^{-\gamma t} X(\tau)$ is independent of $(\omega : X(s), s \leq \tau)$,
Calculate drift:

$$E(X(t) - X(s)|X(s) = x) = (e^{-\gamma|t-s|} - 1)x, \tag{1.1}$$

Calculate diffusion:

$$E((X(t) - X(s))^2|X(s) = x) = 1 - e^{-2\gamma(t-s)} + (e^{-\gamma(t-s)} - 1)x)^2 \tag{1.2}$$

(BTW, jump definition $\lim_{t\to s^+} \frac{1}{t-s} \int_{|y-x|>\epsilon} p(s,x;t,y)dy$)
Over small time interval $[s,t]$, using drift-diffusion information, we see that O-U is related to BM as (to leading order):

$$X(t) - X(s) = -\gamma X(s)(t-s) + \sqrt{2\gamma}(W(t) - W(s)),$$

where $W(t)$ denotes BM; or in differential form:

$$dX = -\gamma X dt + \sqrt{2\gamma} dW.$$

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1.2 Integral Form of SDE

For general SDE of the form:

$$dX_t = a(t, X_t)\, dt + b(t, X_t) dW_t,$$

$dW_t = \xi_t\, dt$, ξ_t white noise, the integral equation is:

$$\begin{aligned} X_t(\omega) = X(t,\omega) &= X_{t_0}(\omega) + \int_{t_0}^{t} a(s, X_s(\omega))\, ds \\ &+ \int_{t_0}^{t} b(s, X_s(\omega))\, dW_s(\omega). \end{aligned} \tag{1.3}$$

dW_s is not a regular function, hence the need to define the last integral, more in general:

$$I(f)(\omega) = \int_0^1 f(s,\omega)\, dW_s(\omega),$$

f is a random function of t (stochastic process).
Naturally, one approximates:

$$I(f)(\omega) \approx \sum_{j=1}^{n} f_j [W_{t_{j+1}} - W_{t_j}],$$

for a partition $0 = t_1 < t_2 < \cdots < t_{n+1} = 1$. *The issue is how to choose* f_j . As both f_j and increment of BM are random, the product is difficult to handle in general ("coupled" or "correlated") unless there is some way to introduce *decoupling* !

Nonanticipating f: let A_t be a sequence of increasing σ-algebra such that W_t is measurable for each $t > 0$, or A_t contains all the events where we observe BM up to time t.

Suppose f is a step function:

$$f(t,\omega) = f_j(\omega),\ t \in [t_j, t_{j+1}].$$

We call f nonanticipating if f_j's are all A_{t_j} measurable or observable by events at or before time t_j. If f is not step function, define f as a limit (in the mean square sense) of such nonanticipating step functions.
With such additional information on f_j, we can take $E[\cdot]$:

$$\begin{aligned} E(f_j[W_{t_{j+1}} - W_{t_j}]) &= E(E(f_j[W_{t_{j+1}} - W_{t_j}]|A_{t_j})) \\ &= E(f_j E([W_{t_{j+1}} - W_{t_j}]|A_{t_j})) = 0. \end{aligned} \tag{1.4}$$

Also:

$$
\begin{aligned}
E(I(f)^2) &= \sum_{j=1}^{n} E(f_j^2 E([W_{t_{j+1}} - W_{t_j}]^2 | A_{t_j})) \\
&= \sum_{j=1}^{n} E(f_j^2)\,(t_{j+1} - t_j), \qquad (1.5)
\end{aligned}
$$

which is a Riemann sum.
If f is not a step function, suppose $E(|f_n - f|^2) \to 0$, Ito showed that $I(f_n)$ has a unique limit, defined as $I(f)$, the Ito stochastic integral.
Properties:
• $I(f)$ is A_1 measurable;
• $E(I(f)) = 0$, $E(I(f)^2) = \int_0^1 E(f(s,\omega))^2\,ds$.
• $I(f)$ is linear in f.

Ito integral defined on $[0,t]$:

$$X_t(\omega) = \int_0^t f(s,\omega)\,dW_s(\omega),$$

is A_t measurable, nonanticipating, mean zero, and:

$$E(X_t^2) = \int_0^t E(f(s,\omega)^2)\,ds.$$

1.3 Sketch of Proof by Ito

Define the space of stochastic process $(\mathcal{L}_T^2)$ by norm,

$$\|f\|_{2,T} = \sqrt{\int_0^T E\left(f(t,\cdot)^2\right) dt}. \qquad (1.6)$$

We denote by $\mathcal{S}_T^2$ the subset of all step functions in $\mathcal{L}_T^2$. Then we can approximate any function in $\mathcal{L}_T^2$ by step functions in $\mathcal{S}_T^2$ to any desired degree of accuracy in the norm. To be specific we have

$\mathcal{S}_T^2$ is dense in $(\mathcal{L}_T^2, \|\cdot\|_{2,T})$, $I(f)$ in $\mathcal{S}_T^2$ is well-defined and satisfies the properties.

Next, for an arbitrary function $f \in \mathcal{L}_T^2$, let $f^{(n)} \in \mathcal{S}_T^2$ a sequence of step functions that,

$$\int_0^T E\left(\left|f^{(n)}(t,\cdot) - f(t,\cdot)\right|^2\right) dt \to 0 \quad \text{as} \quad n \to \infty$$

The last step, we will prove $I\left(f^{(n)}\right)$ is a Cauchy sequence in the Banach space $L^2(\Omega, \mathcal{A}, P)$.

2 Properties of Ito Integrals

We know the Ito integral $I(f)$ is well-defined by passing to limit an approximation:

$$I(f) = \sum_{j=1}^{n} f(t_j, \omega)[W_{t_{j+1}}(\omega) - W_{t_j}(\omega)],$$

the left-hand rule. Such $I(f)$ has nice properties:
• $I(f)$ is A_T measurable, $E(I(f)) = 0$;
• $E(I(f)^2) = \|f\|_{2,T}$, I is linear in f.
It follows:

$$E(I(f)I(g)) = \int_0^T E(f(t,\cdot)g(t,\omega))\, dt.$$

Define Z_t to be the Ito integral when T is set to t, then:

$$E(Z_t - Z_s | A_s) = 0, \tag{2.7}$$

for $s \leq t$. Such a process Z_t is called **martingale**. If $=$ is replaced by $\leq$ ($\geq$), supermartingale (submartingale).
Example: $\xi_n = 1$ if a tossed fair coin is head, otherwise -1. Let $A_n = \sigma(\xi_1, \xi_2, \cdots, \xi_n)$, $n \geq 1$, $X_0 = 0$, $A_0 = \{\varphi, \Omega\}$, $X_n = \xi_1 + \cdots + \xi_n$ is a martingale with respect to A_n. Let g be any convex function, then $g(X_n)$ is a submartingale:

$$E(g(X_n))|A_k) \geq g(E(X_n|A_k)) = g(X_k),\ k \leq n.$$

by Jensen's inequality.

Given Z_t is a mean-square martingale, the maximal martingale inequality holds,

$$P\left(\sup_{t_0 \leq s \leq t} |Z_s| \geq a\right) \leq \frac{1}{a^2} \int_{t_0}^{t} E\left(f(s,\cdot)^2\right) ds,$$

for any $a > 0$, and the Doob inequality holds

$$E\left(\sup_{t_0 \leq s \leq t} |Z_s|^2\right) \leq 4 \int_{t_0}^{t} E\left(f(s,\cdot)^2\right) ds.$$

Note $I(f) - I(f^{(n)})$ is also martingale, and $I(f^{(n)})$ are obviously continuous. Now by carefully select $f^{(n)}$, we can prove,

There is a separable, jointly measurable version of Z_t defined by

$$Z_t(\omega) = \int_{t_0}^{t} f(s,\omega) dW_s(\omega)$$

for $t \in [t_0, T]$ has, **almost surely, continuous sample paths**.

3 Peculiarities of Ito Integrals

Ito integral is different from Riemann integral:

$$\int_0^t W_s(\omega)\, dW_s(\omega) \neq W_t^2(t,\omega)/2,$$

instead:

$$\int_0^t W_s(\omega)\, dW_s(\omega) = W_t^2(t,\omega)/2 - t/2. \tag{3.8}$$

$$\sum_{j=1}^{n} W_{t_j}(W_{t_{j+1}} - W_{t_j}) = W_t^2/2 - \frac{1}{2}\sum_{j=1}^{n}((W_{t_{j+1}} - W_{t_j})^2.$$

The last sum is over i.i.d random variable, hence converging to its mean w.p. 1, that is t.

Next consider chain rule for a composite function $Y_t = U(t, X_t)$, where $dX_t = bdW_t$. The difference is Taylor expanded:

$$\begin{aligned}
\Delta Y_t &= U(t+\Delta t, X_t + \Delta X_t) - U(t, X_t) \\
&= U_t \Delta t + U_X \Delta X_t + \\
&\quad + \frac{1}{2}[U_{tt}(\Delta t)^2 + 2U_{tX}\Delta t \Delta X_t + U_{XX}(\Delta X_t)^2].
\end{aligned} \tag{3.9}$$

The second order terms go to zero if X is a smooth function, however now $E[(\Delta X_t)^2] \approx E[b^2]\Delta t$, behaving like first order term $U_t \Delta t$. *It is this correction that makes the difference.*

Ito formula:

$$dY_t = (U_t + \frac{1}{2}b^2 U_{XX})\, dt + U_X dX_t. \tag{3.10}$$

4 General Ito's Formula

The price to pay for having the nice properties of last section is a different rule for calculus. Let:

$$dX_t = a(t,\omega)\, dt + b(t,\omega)\, dW_t,$$

both e and f nonanticipating; let $U = U(t,x)$ be C^1 in t, C^2 in x. Then:

$$\begin{aligned}
U(t, X_t) &= U(s, X_s) + \int_0^t [U_t + aU_x + b^2 U_{xx}/2](s, X_s)\, ds \\
&\quad + \int_0^t bU_x(s, X_s)\, dW_s,
\end{aligned} \tag{4.11}$$

The operator appearing in the first integral:

$$LU \equiv aU_x + b^2 U_{xx}/2,$$

is dual of the right hand side of Kolmogorov forward equation for transitional probability density, recalling that

$$p_t = -(a(t,x)p)_x \frac{1}{2}(b^2(t,x)p)_{xx},$$

5 Stratonovich Integral

Instead of left hand rule, one could approximate as:

$$S_\lambda(f) = \sum_j [(1-\lambda)f_j + \lambda f_{j+1}][W_{j+1} - W_j],$$

where $\lambda \in [0,1]$, $W_j = W(t_j, \omega)$ etc, resulting integrals are so called (λ)-integrals.
Example:

$$(\lambda)\int_0^T W_t \, dW_t = W_T^2/2 + (\lambda - \frac{1}{2})T.$$

Write $S_\lambda(W_t)$ into two parts:

$$\sum_j W_j[W_{j+1} - W_j] \to W_T^2/2 - T/2,$$

$$\begin{aligned} \sum_j W_{j+1}[W_{j+1} - W_j] &= \sum_j (W_{j+1} - W_j)^2 + W_j[W_{j+1} - W_j] \\ &\to T + W_T^2/2 - T/2. \end{aligned} \tag{5.12}$$

The mid point rule $\lambda = 1/2$ is called Stratonovich, where BM correction vanishes, and standard calculus applies.
Relationship between Ito and Stratonovich ($f = f(W_t)$):

$$\begin{aligned} S_{1/2}(f) &= \sum_j f(W_j)[W_{j+1} - W_j] \\ &+ \frac{1}{2}\sum_j f'(W_j)[W_{j+1} - W_j]^2 + \cdots \\ &\to \int_0^T f(W_t)\, dW_t + \frac{1}{2}\int_0^T f'(W_t)\, dt, \end{aligned} \tag{5.13}$$

in the mean square sense.

$$\int_0^T f(W_t) \circ dW_t = \int_0^T f(W_t)\, dW_t + \frac{1}{2} \int_0^T f'(W_t)\, dt. \tag{5.14}$$

Let $F' = f$, Ito formula gives:

$$F(W_t) - F(W_0) = \int_0^T f(W_t)\, dW_t + \frac{1}{2} \int_0^T f'(W_t)\, dt. \tag{5.15}$$

Hence:

$$\int_0^T f(W_t) \circ dW_t = F(W_t) - F(W_0),$$

as in classical calculus.

6 Stratonovich SDE

$$dX_t = a(t, X_t)dt + b(t, X_t) \circ dW_t, \tag{6.16}$$

or:

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s) \circ dW_s, \tag{6.17}$$

where the stochastic integral is understood as mean square limit of:

$$S_n(\omega) = \sum_{j=0}^{n} b(t_j, (X_{t_j} + X_{t_{j+1}})/2)(W_{t_{j+1}} - W_{t_j}), \tag{6.18}$$

as $k = t/n \to 0$, where $t_j = jk$, $k = t/n$, $j = 0, 1, \cdots, n$.
Consider Stratonovich integral:

$$\int_0^t h(s, X_s) \circ dW_s, \tag{6.19}$$

• Assume for any finite T:

$$\int_0^T E(|h(t, X_t)|^2)\, dt < \infty,$$

then

Theorem 6.1 (I-S integral transform)

$$\int_0^T h(t, X_t) \circ dW_t = \int_0^T h(t, X_t)dW_t + \frac{1}{2} b(t, X_t) h_x(t, X_t) dt. \tag{6.20}$$

Sketch of Proof: let $X_j = X_{t_j}$,

$$\begin{aligned} & h(t_j,(X_j+X_{j+1})/2) - h(t_j,X_j) \\ &= \frac{1}{2}h_x(t_j,\frac{1}{2}((2-\theta_j)X_j+\theta_j X_{j+1}))(X_{j+1}-X_j), \end{aligned}$$

for random numbers θ_j's.

$$\begin{aligned} \Delta X_j &= X_{j+1} - X_j \\ &= a(t_j,X_j)\Delta t + b(t_j,X_j)\Delta W_j + h.o.t, \end{aligned} \tag{6.21}$$

Each term in the Stratonovich sum is:

$$\begin{aligned} & h(t_j,X_j)\Delta W_j + \frac{1}{2}h_x(\theta_j)\Delta X_j \Delta W_j \\ &= h(t_j,X_j)\Delta W_j + \frac{1}{2}h_x(\theta_j)b(t_j,X_j)(\Delta W_j)^2 \\ &+ \frac{1}{2}h_x(\theta_j)a(t_j,X_j)\Delta t \Delta W_j, \end{aligned} \tag{6.22}$$

$$h_x(\theta_j) = h_x(t_j,\frac{1}{2}((2-\theta_j)X_j+\theta_j X_{j+1})).$$

We derive (6.20) using $E((\Delta W_j)^2) = \Delta t$, $E(k\Delta W_j) = 0$, and ignoring higher order terms. It follows from (6.20):

$$E[\int_0^T h(t,X_t)\circ dW_t] = \frac{1}{2}\int_0^T E[b(t,X_t)h_x(t,X_t)]\,dt.$$

Theorem 6.2 (I-S SDE transform) *Let X_t solve Ito SDE:*

$$dX_t = a(t,X_t)dt + b(t,X_t)\,dW_t,$$

then X_t solves the Stratonovich SDE:

$$dX_t = (a(t,X_t) - \frac{1}{2}b(t,X_t)b_x(t,X_t))dt + b(t,X_t)\circ dW_t.$$

7 Application

Consider homogeneous linear Ito SDE:

$$dX_t = aX_t\,dt + bX_t\,dW_t,$$

a, b constant.
S-T transform gives the Stratonovich SDE:

$$dX_t = (a - b^2/2)X_t dt + bX_t \circ dW_t,$$

solution:

$$X_t = X_s \exp\{(a - b^2/2)(t - s) + b(W_t - W_s)\}.$$