Lecture 6: Euler Approximation

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Abstract

Backward and forward representation, strong and weak convergence of Euler approximation.

1 Euler Method: Order of Convergence

SDE:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t,\ t \in (0, T], \tag{1.1}$$

with initial value X_0 at $t = 0$. Discrete times $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N = T$. Denote $\Delta_n = t_{n+1} - t_n$, $\delta = \max \Delta_n$.
Euler approximation:

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta_n + b(t_n, Y_n)(W_{t_{n+1}} - W_{t_n}), \tag{1.2}$$

with $Y_0 = X_0$.
Y_n is A_n measurable.
Connecting the adjacent discrete values Y_n by straight lines form a continuous function $Y(t)$. Pathwise measure of approximation is:

$$\epsilon = E(|X(T) - Y(T)|), \tag{1.3}$$

reduces to deterministic absolute error at $t = T$ if noise is absent. In actual computation, suppose we have N solutions $Y_k(T)$ from N realizations of BM, then ϵ is approximated by:

$$\tilde{\epsilon} = \frac{1}{N}\sum_{k=1}^{N} |X_k(T) - Y_k(T)|. \tag{1.4}$$

It is an amusing fact that $\epsilon \sim O(\delta^{1/2})$ in the stochastic case while $\epsilon \sim O(\delta)$ in the deterministic case. This is analyzed below.

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1.1 Strong Convergence/Consistency

Strong Convergence if:

$$\lim_{\delta\to 0} E(|X(T) - Y_\delta(T)|) = 0. \tag{1.5}$$

Strong Convergence with order $\gamma > 0$*:*

$$E(|X(T) - Y_\delta(T)|) \leq C\delta^\gamma, \tag{1.6}$$

for any $\delta \in [0, \delta_0]$, $\delta_0 > 0$.
Strong Consistency of Discrete Approximation:

$$E\left(|E(\frac{Y_{n+1}^\delta - Y_n^\delta}{\Delta_n}|A_{t_n}) - a(t_n, Y_n^\delta)|^2\right) \leq c(\delta) \to 0, \tag{1.7}$$

and:

$$E\left[\frac{1}{\Delta_n}|Y_{n+1}^\delta - Y_n^\delta - E(Y_{n+1}^\delta - Y_n^\delta|A_{t_n}) - b(t_n, Y_n^\delta)\Delta W_n|^2\right]$$
$$\leq c(\delta) \to 0. \tag{1.8}$$

for all fixed $Y_n^\delta = y$, $n = 0, 1, 2, \cdots$.
For Euler, strong consistency holds with $c(\delta) \equiv 0$.

1.2 Convergence

Consider the autonomous SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{1.9}$$

Theorem 1.1 *A strongly consistent equidistant time discrete approximation* Y_n^δ *of (1.9) with* $Y^\delta(0) = X_0$ *converges strongly to* X*. In particular, Euler method converges strongly with order 1/2.*

Sketch of Proof:

$$Z(t) = \sup_{s\in[0,t]} E(|Y_{n_s}^\delta - X(s)|^2),$$

$$n_s = \max\{n : t_n \leq s\}.$$

$$Z(t) = \sup_{s\in[0,t]} E[|\sum_{n=0}^{n_s-1}(Y_{n+1}^\delta - Y_n^\delta) - \int_0^s a(X_r)ds - \int_0^s b(X_r)dW_r|^2]$$

$$
\begin{aligned}
\leq\ & C_1 \sup_{s\in[0,t]} \Big\{ E[|\sum_{n=0}^{n_s-1}(E(Y^\delta_{n+1}-Y^\delta_n|A_{t_n})-a(Y^\delta_n)\Delta_n)|^2] \\
+\ & E[|\sum_{n=0}^{n_s-1}(Y^\delta_{n+1}-Y^\delta_n-E(Y^\delta_{n+1}-Y^\delta_n|A_{t_n})-b(Y^\delta_n)\Delta W_n)|^2] \\
+\ & E[|\int_0^{t_{n_s}} a(Y^\delta_{n_r})-a(X_r)\,dr|^2]+E[|\int_0^{t_{n_s}} b(Y^\delta_{n_r})-b(X_r)\,dW_r|^2] \\
+\ & E[|\int_{t_{n_s}}^s a(X_s)ds|^2]+E[|\int_{t_{n_s}}^s b(X_s)dW_s|^2]\Big\} \qquad (1.10)
\end{aligned}
$$

by strong consistency and Lipschitz condition:

$$
Z(t) \leq C_2 \int_0^t Z(s)ds + C_3(\delta + c(\delta)), \qquad (1.11)
$$

the last term of (1.10) contributes $O(\delta)$.
Gronwall inequality:

$$
Z(t) \leq C_4(\delta + c(\delta)), \qquad (1.12)
$$

or:

$$
E(|Y^\delta(T) - X(T)|) \leq C_5\sqrt{\delta + c(\delta)}, \qquad (1.13)
$$

strong convergence. For Euler, $c(\delta) = 0$, $\gamma = 1/2$.

2 Backward and Forward Representations

Let $X(t)$ be a diffusion process (solution of SDE) with drift $a(t,x)$, diffusion $b(t,x)$:

$$
dX_t = adt + bdW_t,
$$

consider the conditional expectation ($s < t$):

$$
E(f(X_t)|X_s = x) = \int f(y)\,p(s,x;t,y)\,dy, \qquad (2.14)
$$

where $p(s,x;t,y)$ is the transition probability density function from (s,x) to (t,y). As a function of (s,x), p satisfies the *backward equation*:

$$
p_s + \frac{1}{2}b^2(s,x)p_{xx} + a(s,x)p_x = 0. \qquad (2.15)
$$

Hence $u(s,x) = E(f(X_t)|X_s = x)$ solves (2.15) with final condition $u(t,x) = f(x)$.
For the forward representation, consider the *Autonomous case*, $a = a(x)$, $b = b(x)$. Then $p(s,t;x,y) = p(t-s;x,y)$, $p_s = -p_t$,

$$
p_t = \frac{1}{2}b^2(x)p_{xx} + a(x)p_x,\ t > s, \qquad (2.16)
$$

$p(t;x,y) \to \delta(y-x)$, as $t \to 0+$. The transition probability density becomes fundamental solution of parabolic equation (2.16). As a function of (t,x),

$$v(t,x) = E(f(X_t)|X_s = x) = \int f(y)\, p(t-s;x,y)\, dy, \tag{2.17}$$

solves:

$$v_t = \frac{1}{2} b^2(x) v_{xx} + a(x) v_x, \tag{2.18}$$

with initial data $v(s,x) = f(x)$.

Feynman-Kac Formula Eq. (2.17) is a probabilistic representation formula of PDE (2.18). It can be generalized to include a lower order (potential, V) term as in Eqn:

$$w_t = \frac{1}{2} b^2(x) w_{xx} + a(x) w_x + V(x) w, \; t > 0, \tag{2.19}$$

initial data: $w(0,x) = f(x)$. Then,

$$w(t,x) = E\left[\exp\{\int_0^t V(X(\tau))\, d\tau\} f(X(\tau))\right], \tag{2.20}$$

Rmk: If the diffusion $b(x) \equiv 0$, F-K formula reduces to a solution formula of first order hyperbolic eqn by the method of characteristics.

To derive (2.20), let:

$$T_t f = E\left[\exp\{\int_0^t V(X(\tau))\, d\tau\} f(X(\tau))\right],$$

a linear bounded (non-negative) operator on the space of bounded continuous functions. Note:

$$\exp\{\int_0^t V(X_s) ds\} = 1 + \int_0^t V(X_s) ds + o(t),$$

as $t \to 0+$. We have for any $f(x)$ in the domain of T_t:

$$\begin{aligned} \frac{T_t f(x) - f(x)}{t} &= \frac{1}{t}\left(E[f(X_t) e^{\int_0^t V(X_s) ds}] - f(x)\right) \\ &= \frac{1}{t}(E[f(X_t)] - f(x)) + \frac{1}{t} E[f(X_t) \int_0^t V(X_s) ds] \\ &\to (b^2(x) f_{xx}/2 + a(x) f_x) + V(x) f. \end{aligned} \tag{2.21}$$

We have used (2.16) for the limit of first term.

To generalize F-K to nonautonomous case, we should notice trick in (2.16) no longer works. Treat t as a parameter,

$$\begin{aligned} dX_s^{t,x} &= a(t_s^{t,x}, X_s^{t,x}) ds + b(t_s^{t,x}, X_s^{t,x}) dW_s, \\ dt_s^{t,x} &= -ds, \end{aligned} \tag{2.22}$$

$X_0^{t,x} = x$, $t_0^{t,x} = t$, symmetrically extending a, b: $a(-\tau, x) = a(\tau, x)$ etc. *View (2.22) as a diffusion process on* $(t, x) \in R^2$ *with time* s. Eqs (2.22) are autonomous, and define a Markov process $(t_s^{t,x}, X_s^{t,x}, P)$. We then apply F-K (2.20). The result is:

$$w(t, x) = Ef(X_t^{t,x}) \exp\{\left[\int_0^t V(t-s, X_s^{t,x}) ds\right]\}, \tag{2.23}$$

solves eqn:

$$w_t = \frac{1}{2} b^2(t, x) w_{xx} + a(t, x) w_x + V(t, x) w, \tag{2.24}$$

$w(0, x) = f(x)$.

All results generalize to higher space dimensions.

3 Weak Consistency: Definition and Examples

A discrete SDE approximation $Y^\delta(t)$ is called *converging weakly* to $X(t)$ at $t = T$ if:

$$\lim_{\delta \to 0} |E(g(X(T))) - E(g(Y^\delta(T)))| = 0, \tag{3.25}$$

for any $g \in \mathcal{C}$, $\mathcal{C}$ a class of smooth test functions. One example of $\mathcal{C}$ is all polynomials, then (3.25) is same as convergence of all moments of solutions. As before, discrete times $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N = T$, $\Delta_n = t_{n+1} - t_n$, $\delta = \max \Delta_n$.
Convergence is order $\beta > 0$ if:

$$|E(g(X(T))) - E(g(Y^\delta(T)))| \leq C\delta^\beta, \tag{3.26}$$

for small δ.
Later we will see that Euler method is weakly convergent of order $\beta = 1$, while it is order 1/2 strong convergent (pathwise).

The discrete approximation is *weakly consistent* if

$$E\left(\left|E\left(\frac{Y_{n+1}^\delta - Y_n^\delta}{\Delta_n} | A_{t_n}\right) - a(t_n, Y_n^\delta)\right|^2\right) \leq c(\delta) \to 0, \tag{3.27}$$

same as in strong consistency, and:

$$E\left[\left|E\left(\frac{1}{\Delta_n}(Y_{n+1}^\delta - Y_n^\delta)^2 | A_n\right) - b^2(t_n, Y_n^\delta)\right|^2\right] \leq c(\delta) \to 0. \tag{3.28}$$

for all fixed $Y_n^\delta = y$, $n = 0, 1, 2, \cdots$.
For Euler, weak consistency holds. Moreover, some modified Euler like:

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta_n + b(t_n, Y_n)\xi_n (\Delta_n)^{1/2}, \tag{3.29}$$

where ξ_n independent two point r.v., $P(\xi_n = \pm 1) = 1/2$, is weakly convergent, not strongly convergent.

4 Consistency implies Convergence

Consider the autonomous SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{4.30}$$

a, b, smooth, with polynomial growth.

Theorem 4.1 *Consider equidistant time weakly consistent discrete approximation Y_n^δ of (4.30) with $Y^\delta(0) = X_0$ so that:*

$$E(\max_n |Y_n^\delta|^{2q}) \le K(1 + E(|X_0|^{2q})), \tag{4.31}$$

for $q = 1, 2, \cdots$, and:

$$E(|Y_{n+1}^\delta - Y_n^\delta|^6) \le c(\delta)\Delta_n, \quad c(\delta) = o(\delta), \tag{4.32}$$

for any $n = 0, 1, 2, \cdots$. Then Y_n^δ converges weakly to $X(t)$.

Sketch of Proof: Write $Y(t) = Y^\delta(t)$.
Use fact:

$$u(s, x) = E(g(X_T)|X_s = x), \tag{4.33}$$

solves backward equation:

$$u_s + Lu = u_s + au_x + \frac{b^2}{2}u_{xx} = 0, \tag{4.34}$$

and:

$$u(T, x) = g(x). \tag{4.35}$$

Denote by $X_t^{s,x}$ solution of:

$$X_t^{s,x} = x + \int_s^t a(X_r^{s,x})dr + \int_s^t b(X_r^{s,x})dW_r. \tag{4.36}$$

A key observation by Ito formula and (4.34) :

$$E(u(t_{n+1}, X_{t_{n+1}}^{t_n,x}) - u(t_n, x)|A_n) = 0, \tag{4.37}$$

By eqns (4.33)-(4.35), write:

$$\begin{aligned} H &= |E(g(Y(T))) - E(g(X(T)))| \\ &= |E(u(T, Y(T)) - E(u(T, X(T))| \\ &= |E(u(T, Y(T)) - u(0, Y_0))| \\ &= |E(\sum_{n=0}^{n_T - 1} u(t_{n+1}, Y_{n+1}) - u(t_n, Y_n))|. \end{aligned} \tag{4.38}$$

By (4.37):

$$\begin{aligned} H &= |E(\sum [u(t_{n+1}, Y_{n+1}) - u(t_n, Y_n) \\ &\quad -(u(t_{n+1}, X_{t_{n+1}}^{t_n, Y_n}) - u(t_n, X_{t_n}^{t_n, Y_n}))])| \\ &= |E(\sum [u(t_{n+1}, Y_{n+1}) - u(t_{n+1}, Y_n) \\ &\quad -(u(t_{n+1}, X_{t_{n+1}}^{t_n, Y_n}) - u(t_{n+1}, Y_n))])| \end{aligned}$$

Taylor expand in x:

$$\begin{aligned} H &= |E(\sum u_x[(Y_{n+1} - Y_n) - (X_{t_{n+1}}^{t_n, Y_n} - Y_n)] \\ &\quad + \frac{1}{2} u_{xx}[(Y_{n+1} - Y_n)^2 - (X_{t_{n+1}}^{t_n, Y_n} - Y_n)^2] \\ &\quad + O(|X_{t_{n+1}}^{t_n, Y_n} - Y_n|^3) + |Y_{n+1} - Y_n|^3)| \end{aligned} \tag{4.39}$$

u_x, u_{xx} evaluated at (t_{n+1}, Y_n).

Higher Moments Estimate of SDE (augmented, Theorem 4.5.4 in KL's book)
Suppose that conditions in lecture 5 hold and that

$$E\left(|X_{t_0}|^{2n}\right) < \infty$$

for some integer $n \geq 1$. Then the solution X_t satisfies

$$E\left(|X_t|^{2n}\right) \leq \left(1 + E\left(|X_{t_0}|^{2n}\right)\right) e^{C(t-t_0)}$$

and

$$E\left(|X_t - X_{t_0}|^{2n}\right) \leq D\left(1 + E\left(|X_{t_0}|^{2n}\right)\right) (t - t_0)^n \, e^{C(t-t_0)}$$

$$\begin{aligned} H &\leq C \sum E(|u_x| \Big| E((Y_{n+1} - Y_n) - (X_{t_{n+1}}^{t_n, Y_n} - Y_n)|A_n) \Big| \\ &+ \frac{1}{2}|u_{xx}||E((Y_{n+1} - Y_n)^2 - (X_{t_{n+1}}^{t_n, Y_n} - Y_n)^2|A_n)| \\ &\quad + O(\delta^{3/2} + \delta^{1/2}\sqrt{c(\delta)}) \\ &\leq C \sum \delta E^{1/2}(|E(\frac{Y_{n+1} - Y_n}{\delta}|A_n) - a(t_n, Y_n)|^2) \\ &+ \delta E^{1/2}(|E(\frac{(Y_{n+1} - Y_n)^2}{\delta}|A_n) - b^2(t_n, Y_n)|^2) \\ &+ O(\delta^{3/2} + \delta^{1/2}\sqrt{c(\delta)}) \\ &\leq C \sum \delta\sqrt{c(\delta)} + O(\delta^{3/2} + \delta^{1/2}\sqrt{c(\delta)}) \\ &= O(\sqrt{c(\delta)} + \delta^{1/2} + \sqrt{c(\delta)/\delta}) \to 0. \end{aligned} \tag{4.40}$$