Math 18500 Week 3: Systems of First Order Linear Equations

Linear Autonomous Systems. We have seen that every system of first order autonomous equations

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

can be solved, at least in theory. But in practice, the method we have outlined is badly ineffective. We will now describe a more effective method for solving systems of the following general form:

$$\frac{dx}{dt} = a_{11}x + a_{12}y + b_1$$
$$\frac{dy}{dt} = a_{21}x + a_{21}y + b_2$$

A system of this form is said to be a *linear*, *autonomous* system of equations. Since we will only be considering autonomous systems, we will often drop the adjective "autonomous". However, it is possible to consider non-autonomous linear systems as well (in which case the coefficients a_{ij} and b_i would be functions of t, rather than constants).

Any linear system can always be written in *matrix form*, like this:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Often we will also use the following shorthand:

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}\ ,\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\ ,\quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\ ,\quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

It will also be useful to distinguish between two separate kinds of linear systems. If

$$\vec{b} = \vec{0}$$

we say that the system is **homogeneous**, and if

$$\vec{b} \neq \vec{0}$$

we say that the system is **inhomogeneous**.

For example, the linear system

$$\begin{cases} \dfrac{dx}{dt} & = & x + 3y \\ \dfrac{dy}{dt} & = & x - y \end{cases}$$

is homogeneous, because it can be written in matrix form without a constant term:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Likewise, the linear system

$$\begin{cases} \dfrac{dx}{dt} & = & x + 3y + 1 \\ \dfrac{dy}{dt} & = & x - y - 1 \end{cases}$$

is inhomogeneous, because its matrix form has a nonzero constant term:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

There are two important initial observations that must be made about linear systems:

(1) Suppose that $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions of a homogeneous system:

$$\frac{d\vec{x}_i}{dt} = A\vec{x}_i\ ,\quad i = 1, 2.$$

If c_1 and c_2 are arbitrary constants, and

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t),$$

then $\vec{x}(t)$ is a solution of the same homogeneous system:

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

In other words, *solutions of homogeneous equations can be superimposed.*

(2) Suppose that $\vec{x}_p(t)$ is a *particular* solution of an inhomogeneous system,

$$\frac{d\vec{x}_p}{dt} = A\vec{x}_p + \vec{b}.$$

If $\vec{x}_h(t)$ is any solution of the corresponding *homogeneous* equation,

$$\frac{d\vec{x}_h}{dt} = A\vec{x}_h.$$

and if

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_h(t),$$

then $\vec{x}(t)$ is a solution of the original inhomogeneous equation:

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$

Observation (1) is referred to as the **superposition principle**. The superposition principle is the key to solving linear equations - you will see it recur over and over again in this course (and 186) in various guises.

To prove the superposition principle, we just expand both sides:

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\left[c_1\vec{x}_1 + c_2\vec{x}_2\right] = c_1\frac{d\vec{x}_1}{dt} + c_2\frac{d\vec{x}_2}{dt}$$
$$A\vec{x} = A\left[c_1\vec{x}_1 + c_2\vec{x}_2\right] = c_1A\vec{x}_1 + c_2A\vec{x}_2$$

By assumption, since

$$\frac{d\vec{x}_i}{dt} = A\vec{x}_i,$$

the two sides of the equation

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

are equal, i.e. $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ is a solution of the equation.

Observation (2) can be proved in exactly the same way. Expanding both sides, we get

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\left[\vec{x}_p + \vec{x}_h\right] = \frac{d\vec{x}_p}{dt} + \frac{d\vec{x}_h}{dt}$$
$$A\vec{x} + \vec{b} = A\left[\vec{x}_p + \vec{x}_h\right] + \vec{b} = \left(A\vec{x}_p + \vec{b}\right) + A\vec{x}_h$$

These are equal due to the assumptions

$$\frac{d\vec{x}_p}{dt} = A\vec{x}_p + \vec{b}$$

and

$$\frac{d\vec{x}_h}{dt} = A\vec{x}_h.$$

To exploit observation (2), we need to find a particular solution $\vec{x}_p$. In most cases, it's easy to find *one* such solution - we can just look for *equilibrium points*! This is something we should always do in any case, as a first step towards analyzing a system of first order equations.

Given a linear inhomogeneous equation

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b},$$

we can find all equilibrium points by looking for solutions $\vec{x}(t) = \vec{x}_{eq}$ such that

$$\frac{d\vec{x}_{eq}}{dt} = 0$$

This is equivalent to solving the matrix equation

$$A\vec{x}_{eq} + \vec{b} = 0,$$

and as long as A is an *invertible* matrix, this equation will always have *exactly one solution.* [1]

For example, to find the equilibrium solution of the equation

$$\begin{cases} \frac{dx}{dt} & = & x + 3y + 1 \\ \frac{dy}{dt} & = & x - y - 1 \end{cases}$$

We just need to solve the system of equations

$$\begin{cases} x_{eq} + 3y_{eq} & = & -1 \\ x_{eq} - y_{eq} & = & 1 \end{cases}$$

This can either be done by hand, or by converting the system of equations to a matrix equation,

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{eq} \\ y_{eq} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and multiplying on both sides by an inverse matrix:

$$\begin{bmatrix} x_{eq} \\ y_{eq} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{-1}{4} \begin{bmatrix} -1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

If the matrix A is not invertible, there may not be any equilibrium solutions - we'll ignore this case.

Assuming we can find an equilibrium solution, the general solution of the inhomogeneous system

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$

will be given by

$$\vec{x}(t) = \vec{x}_{eq} + \vec{x}_h,$$

where $\vec{x}_h$ is a solution of the homogeneous system,

$$\frac{d}{dt}\vec{x}_h = A\vec{x}_h.$$

Therefore, the rest of this week will be focused on solving homogeneous equations. Our strategy will be to apply the superposition principle - we will obtain solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$ by "guessing", and then we will superimpose them to obtain a general solution of the homogeneous equation,

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t).$$

In cases where we want to solve an initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}\ ,\ \ \vec{x}(0) = \vec{x}_0,$$

we will write the general solution of the *inhomogeneous* equation in the form

$$\vec{x}(t) = \vec{x}_{eq} + c_1\vec{x}_1(t) + c_2\vec{x}_2(t),$$

and we will determine c_1 and c_2 by setting $t = 0$ and solving the resulting system of equations:

$$\vec{x}_0 = \vec{x}_{eq} + c_1\vec{x}_1(0) + c_2\vec{x}_2(0).$$

As long as the vectors $\vec{x}_1(0)$ and $\vec{x}_2(0)$ are a 2-dimensional *basis*, this will always be possible.

[1] An example of a system which does not have an equilibirum solution is $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = x$. It's a nice exercise to find all solutions of this system and plot them in the xy plane.

Homogeneous Linear Systems: Real Eigenvalue Case. In this section we will begin to address the problem of finding two solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$ of a linear, homogeneous, autonomous system

$$\frac{d\vec{x}}{dt} = A\vec{x},$$

such that $\vec{x}_1(0)$ and $\vec{x}_2(0)$ are a two-dimensional basis.

As a concrete example, consider the homogeneous system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

To understand what the solutions of this system look like, we first compute the slope of its velocity field,

$$\frac{dy}{dx} = \frac{-2x+y}{x-2y},$$

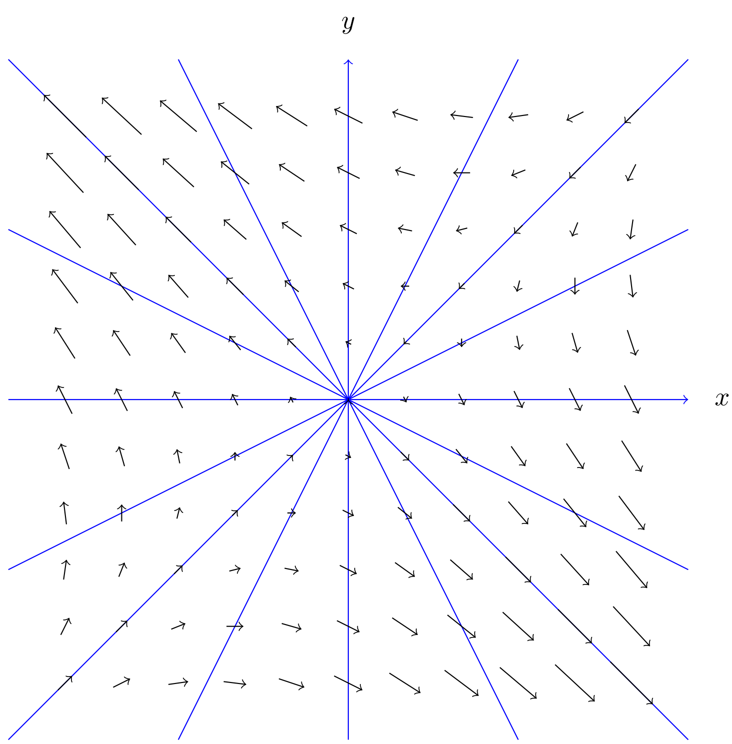
and we write down its isocline equations:

$$\frac{-2x+y}{x-2y} = C = \text{constant}.$$

Simplifying this equation, we find that all isoclines are *lines through the origin*:

$$-2x + y = C(x-2y) \implies (1+2C)y = (2+C)x.$$

We can see this effect when we plot the velocity field:



Notice that there are two isoclines which are everywhere *parallel* to the velocity field. One of these is the isocline with slope 1:

$$C = 1 \implies (1+2)y = (2+1)x \implies y = x.$$

Geometrically, the fact that the velocity field is parallel to this line tells us that our system of equations has a special solution of the form

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = f_1(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_1(t) \end{bmatrix}.$$

To solve for the unknown function $f_1(t)$, we can substitute into both sides of our system:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} f_1'(t) \\ f_1'(t) \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_1(t) \end{bmatrix} = \begin{bmatrix} -f_1(t) \\ -f_1(t) \end{bmatrix}.$$

From this we see that $f_1(t)$ must satisfy the differential equation

$$f_1'(t) = -f_1(t),$$

and therefore must take the general form

$$f_1(t) = c_1 e^{-t}$$

for some constant c_1.

An important thing to notice here is that

$$\lim_{t\to\infty}\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \lim_{t\to\infty}\begin{bmatrix} c_1e^{-t} \\ c_1e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is consistent with the velocity field: along the line $y = x$, the velocity field points directly inward, towards the origin.

By similar reasoning, we can find a solution which travels along the isocline with slope $C = -1$:

$$C = -1 \implies (1-2)y = (2-1)x \implies y = -x$$

This solution takes the form

$$\vec{x}_2(t) = f_2(t)\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and the function $f_2(t)$ can be found using the same process:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} f_2'(t) \\ -f_2'(t) \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} f_2(t) \\ -f_2(t) \end{bmatrix} = \begin{bmatrix} 3f_2(t) \\ -3f_2(t) \end{bmatrix}.$$

From this we see that $f_2(t)$ must satisfy the differential equation

$$f_2'(t) = 3f_2(t),$$

and therefore must take the general form

$$f_2(t) = c_2 e^{3t}$$

for some constant c_2. This time, we have

$$\lim_{t\to\infty} f_2(t) = \infty.$$

Again, this is consistent with the velocity field, which points away from the origin on the line $y = -x$.

How can we generalize these solutions to other linear homogeneous systems? The answer is that we must look for solutions that start at points (x_0, y_0) where the position vector,

$$\vec{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is *parallel* to the velocity specified by the velocity field:

$$A\vec{v} = A\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

To make $A\vec{v}$ and $\vec{v}$ parallel, $\vec{v}$ must satisfy the *eigenvector equation*:

$$A\vec{v} = \lambda\vec{v}.$$

Given any eigenvector $\vec{v}$, we can immediately find a solution of our system, in the form

$$\vec{x}(t) = f(t)\vec{v},$$

and we can also immediately solve for $f(t)$ by substituting into the equation

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

Explicitly, we find that

$$f'(t)\vec{v} = Af(t)\vec{v} = f(t)\lambda\vec{v},$$

and therefore

$$f(t) = ce^{\lambda t}$$

for some constant c.

In cases where A is a 2×2 matrix with a *real eigenbasis*, this idea produces two solutions,

$$\vec{x}_1(t) = c_1 e^{\lambda_1 t}\vec{v}_1 \ , \ \ \vec{x}_2(t) = c_2 e^{\lambda_2 t}\vec{v}_2.$$

and we can obtain a general solution by *superimposing* them:

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$

For example, suppose we want to find a solution of the initial value problem

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \ , \ \ x(0) = 2, \ y(0) = 1.$$

For this system, we have already obtained the two special solutions

$$\vec{x}_1(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ \ \vec{x}_2(t) = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

which correspond to the following solutions of the eigenvector equation:

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ \ \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the general solution is

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Setting $t = 0$, and imposing our initial conditions, we obtain

$$\vec{x}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

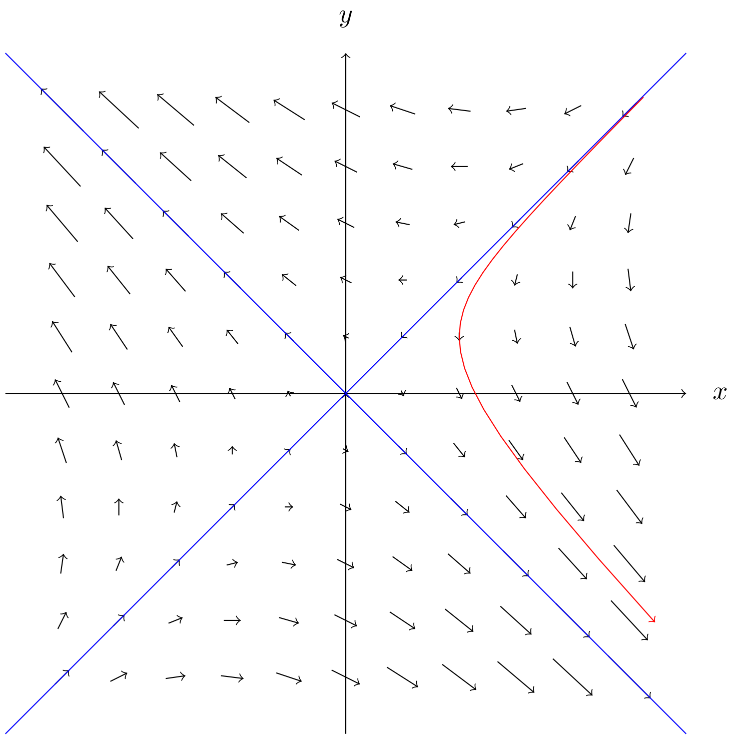
You can verify that the solution of this system of linear equations is

$$c_1 = \frac{3}{2} \ , \ \ c_2 = \frac{1}{2}.$$

Therefore, the solution of the initial value problem is

$$\vec{x}(t) = \frac{3}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Here is a plot of the solution:



You can see that the solution starts off asymptotic to the line $y = x$, in the limit as $t \to -\infty$. It's a bit harder to see from the picture, but in the limit as $t \to \infty$ it is asymptotic to the line $y = -x$. The reason for this is that

$$\lim_{t\to\infty} \frac{3}{2}e^{-t}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

and

$$\lim_{t\to-\infty} \frac{1}{2}e^{3t}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

so in each of these limits only one of the two terms dominates the behavior of $\vec{x}(t)$.

The reason why it takes the solution much longer to approach the line $y = -x$ is that the exponential

$$e^{-t}$$

decays to 0 at a slower rate in the limit $t \to \infty$ than the exponential

$$e^{3t}$$

decays to 0 in the limit as $t \to -\infty$.

The relative rates of exponential decay actually become very important for systems where the eigenvalues of the coefficient matrix have the same sign. For example, consider the initial value problem

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\ ,\ \ x(0) = 2\ ,\ \ y(0) = 1$$

The coefficient matrix for this system,

$$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

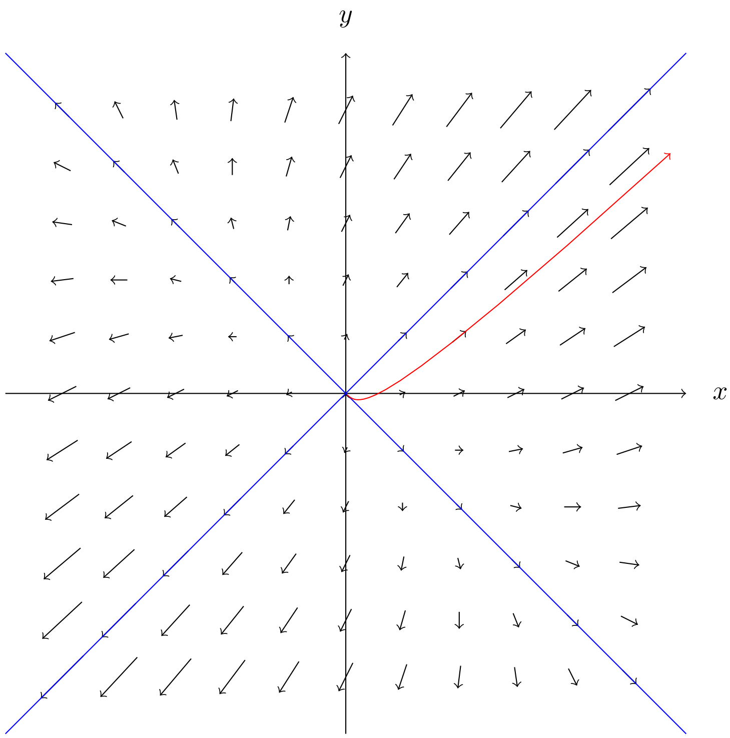
has the same eigenvectors as in our previous example, but the eigenvalues have the same sign:

$$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 3\cdot\begin{bmatrix}1\\1\end{bmatrix}\ ,\quad \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 1\cdot\begin{bmatrix}1\\-1\end{bmatrix}$$

By applying the methods above, you can find the solution:

$$\vec{x}(t) = \frac{3}{2}e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}e^{t}\begin{bmatrix}1\\-1\end{bmatrix}.$$

Here a plot of the solution, together with the velocity field of the system:



In the limit as $t \to \infty$, the solution curve is nearly parallel to the line $y = x$. This is because the exponential

$$e^{3t}$$

grows at a faster rate than the exponential

$$e^{t},$$

so the first term dominates the solution in the limit as $t \to \infty$.

Similarly, in the limit as $t \to -\infty$, the exponential

$$e^{t}$$

decays slower than the exponential

$$e^{3t},$$

so as $t \to -\infty$, the second term dominates. This is also reflected in the plot: as the solution approaches the origin, it does so in a direction which is tangent to the line $y = -x$.

Homogeneous Linear Systems: Complex Eigenvalue Case. In 183 you learned that not every 2×2 matrix has a real eigenbasis. For example, consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

If we compute its characteristic polynomial,

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 4$$

we find that it has two complex roots:

$$(1-\lambda)^2 + 4 = 0 \implies \lambda = 1 \pm 2i.$$

Therefore, A has no real eigenvalues (and certainly no real eigenbasis).

But it does have *complex* eigenvalues! By definition, a complex eigenvalue of a matrix A is a complex number

$$\lambda = p + iq$$

such that there exists a complex vector

$$\vec{v} = \begin{bmatrix} v_1 + w_1 i \\ v_2 + w_2 i \end{bmatrix},$$

which satisfies

$$A\vec{v} = \lambda\vec{v} = (p + iq) \begin{bmatrix} v_1 + w_1 i \\ v_2 + w_2 i \end{bmatrix}.$$

If λ is any complex root of the characteristic polynomial of A, it is always possible to find a complex eigenvector with eigenvalue λ, using the same methods we use to find real eigenvectors.

For example, we can find a complex eigenvector of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix},$$

with eigenvalue

$$\lambda = 1 + 2i,$$

by solving the equation

$$\begin{bmatrix} 1-(1+2i) & -2 \\ 2 & 1-(1+2i) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One solution of the equation is

$$\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

and this is a complex eigenvector for A.

We can use this complex eigenvector to construct a solution of the equation

$$\frac{d\vec{x}}{dt} = A\vec{x},$$

in exactly the same way that we used real eigenvectors to construct solutions. The main difference is that the solution is a complex vector:

$$\vec{x}(t) = e^{\lambda t}\vec{v} = e^{(1+2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

To understand why this is a valid solution, look at what happens when we plug it back in to the equation:

$$\frac{d}{dt}\left[e^{\lambda t}\vec{v}\right] = \lambda e^{\lambda t}\vec{v} = e^{\lambda t}\lambda\vec{v} = e^{\lambda t}A\vec{v} = Ae^{\lambda t}\vec{v}$$

If λ was a real number and $\vec{v}$ was a real vector, these steps would show that $e^{\lambda t}\vec{v}$ was a solution. But there is one step which requires additional verification when λ is complex:

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}.$$

To prove this identity, first expand λ into its real and imaginary parts:

$$\frac{d}{dt}e^{(p+iq)t} = (p+iq)e^{(p+iq)t}.$$

Then expand both sides of the equation into *their* real and imaginary parts:

$$\frac{d}{dt}\left[e^{pt}\cos(qt) + ie^{pt}\sin(qt)\right] = \frac{d}{dt}\left[e^{pt}\cos(qt)\right] + i\frac{d}{dt}\left[e^{pt}\sin(qt)\right]$$

$$(p+iq)\left[e^{pt}\cos(qt) + ie^{pt}\sin(qt)\right] = \left[pe^{pt}\cos(qt) - qe^{pt}\sin(qt)\right] + i\left[pe^{pt}\sin(qt) + qe^{pt}\cos(qt)\right]$$

The two expressions are equal, by virtue of the product rule.

Now let's return to the equation we were trying to solve,

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

We have found a complex solution, but this isn't very satisfying - what we actually want are *real* solutions.

Fortunately, any complex solution leads to not only one but *two* real solutions!

To see why, we can expand our complex solution into its real and imaginary parts:

$$\vec{x}(t) = e^{(1+2i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^{t}\cos(2t) \\ e^{t}\sin(2t) \end{bmatrix} + i\begin{bmatrix} e^{t}\sin(2t) \\ -e^{t}\cos(2t) \end{bmatrix} = \vec{x}_1(t) + i\vec{x}_2(t)$$

We know that this is a solution of the equation

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

Making the substitution $\vec{x}(t) = \vec{x}_1(t) + i\vec{x}_2(t)$, we find that

$$\frac{d\vec{x}_1}{dt} + i\frac{d\vec{x}_2}{dt} = A\vec{x}_1 + iA\vec{x}_2,$$

and comparing the real and imaginary parts of both sides,

$$\frac{d\vec{x}_1}{dt} = A\vec{x}_1\ ,\quad \frac{d\vec{x}_2}{dt} = A\vec{x}_2$$

we find that *both the real and imaginary parts of the complex solution are real solutions*!

In conclusion, we obtain two real solutions,

$$\vec{x}_1(t) = \begin{bmatrix} e^{t}\cos(2t) \\ e^{t}\sin(2t) \end{bmatrix}$$

and

$$\vec{x}_2(t) = \begin{bmatrix} e^{t}\sin(2t) \\ -e^{t}\cos(2t) \end{bmatrix}.$$

These solutions can be used to generate arbitrary real solutions, by taking (real) linear combinations:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

If we want a solution with prescribed initial values, we can set $t = 0$, and solve the equation

$$\vec{x}(0) = c_1\vec{x}_1(0) + c_2\vec{x}_2(0)$$

for c_1 and c_2.

To complete this story, you need to understand how to *visualize* solutions like this. In general, if

$$\lambda = p + iq\ ,\quad q > 0$$

is a complex eigenvalue of a matrix A, then all real solutions of the equation

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

will take the general form

$$\vec{x}(t) = e^{pt}\cos(qt)\vec{v}_1 + e^{pt}\sin(qt)\vec{v}_2,$$

where $\vec{v}_1$ and $\vec{v}_2$ are real vectors. It's helpful to plot these trajectories in the special case

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this case, we have

$$\vec{x}(t) = e^{pt} \begin{bmatrix} \cos(qt) \\ \sin(qt) \end{bmatrix}.$$

No matter the value of p, the curve will "spiral" around the origin, crossing the x axis at regular intervals with period

$$T = \frac{2\pi}{q}.$$

The distance from the origin is controlled by the factor e^{pt}, in a way which depends on the sign of p:

(1) If $p < 0$, then the solution will spiral *towards* the origin, since

$$\lim_{t\to\infty} e^{pt} = 0$$

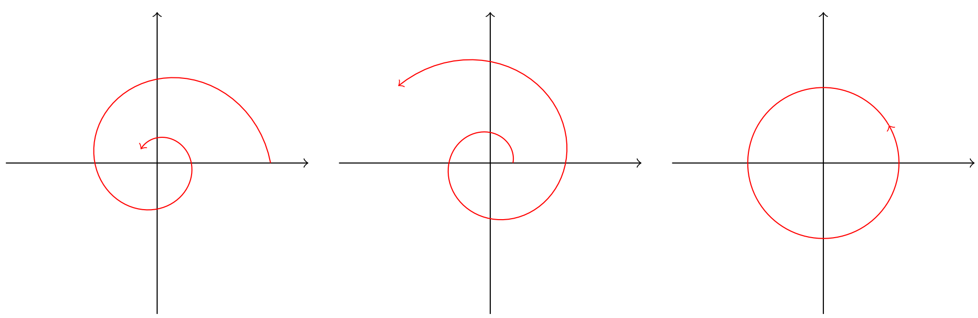
(2) If $p > 0$, the solution will spiral *away* from the origin, since

$$\lim_{t\to\infty} e^{pt} = \infty$$

(3) If $p = 0$, the solution will move around the origin in a circle, since

$$e^{0t} = 1$$

Here are pictures of all three cases:



$p < 0$ $\qquad$ $p > 0$ $\qquad$ $p = 0$

We can plot the solutions in the general case by applying a *linear transformation* to each of the standard solutions above.

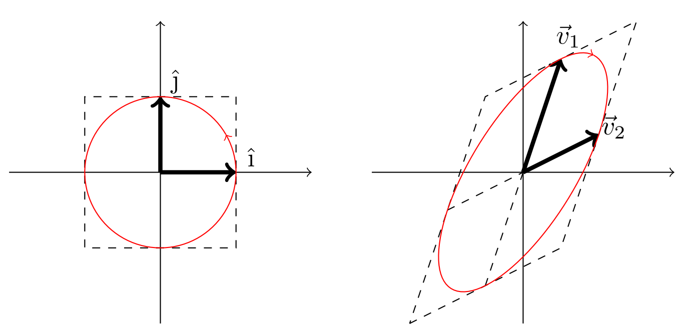
For example, to sketch the trajectory

$$\vec{x}(t) = \cos(qt)\vec{v}_1 + \sin(qt)\vec{v}_2,$$

we would take the standard trajectory

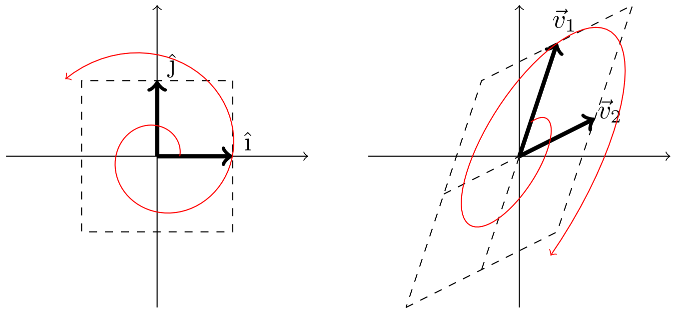
$$\vec{x}_0(t) = \cos(qt)\hat{\imath} + \sin(qt)\hat{\jmath},$$

and apply the following transformation:



To guide the transformation, we draw a square with sides parallel to the x and y axes. When we transform this square, it becomes a parallelogram with sides parallel to $\vec{v}_1$ and $\vec{v}_2$. The circle inscribed in the square becomes an *ellipse* inscribed in the parallelogram!

The pictures when $p \neq 0$ is similar. A spiral aligned with the x and y axes gets transformed to a spiral aligned with the vectors $\vec{v}_1$ and $\vec{v}_2$:



Notice that in both cases, the *orientation* of the spiral changed, from clockwise to counterclockwise. In general, to determine the orientation of the spiral, we need to draw a few velocity vectors.

For example, consider the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

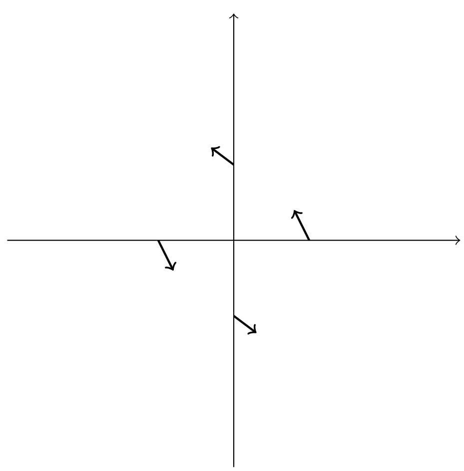
If we calculate the characteristic polynomial of its matrix, we get

$$\det\begin{bmatrix} -1-\lambda & -4 \\ 2 & 3-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 5 = (\lambda - 1)^2 + 2^2.$$

This has complex roots

$$\lambda = 1 \pm 2i,$$

and therefore the solutions will spiral outward (since the real part of the eigenvalue is $p = 1 < 0$). To determine the orientation of the spirals, we can compute the velocity vectors at the points $(\pm 1, 0)$ and $(0, \pm 1)$ and draw them:



Once we draw these, it becomes clear that the spirals must be going counterclockwise.

It's usually pretty difficult to make accurate sketches of the solutions by hand, but it's not hard to get a rough idea of what they look like!

Stability. In summary, we have seen that solutions of linear autonomous systems

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$

can always be written in the form

$$\vec{x}(t) = \vec{x}_{eq} + c_1\vec{x}_1(t) + c_2\vec{x}_2(t),$$

where $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions of the associated homogeneous system

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

This is valid in all cases, as long as an equilibrium solution exists.

In the case where A has two distinct real eigenvalues, the solutions $\vec{x}_i(t)$ are given by

$$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1\ ,\ \ \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2,$$

where λ_1 and λ_2 are the real eigenvalues, and $\vec{v}_1$ and $\vec{v}_2$ are the real eigenvectors.

In the case where A has two complex eigenvalues, they are given by

$$\vec{x}_1(t) = e^{pt}(\cos(qt)\vec{v}_1 - \sin(qt)\vec{v}_2)\ ,\ \ \vec{x}_2(t) = e^{pt}(\sin(qt)\vec{v}_1 + \cos(qt)\vec{v}_2)$$

where $\vec{v}_1$ and $\vec{v}_2$ are the real and imaginary parts of a complex eigenvector

$$\vec{v} = \vec{v}_1 + i\vec{v}_2$$

with eigenvalue

$$\lambda = p + iq.$$

There is also a third case: A could have a *repeated* real eigenvalue. In this case it is always possible to find solutions of the form

$$\vec{x}_1(t) = e^{\lambda t}\vec{v}_1\ ,\ \ \vec{x}_2(t) = e^{\lambda t}\vec{v}_2 + te^{\lambda t}\vec{v}_3$$

where $\vec{v}_1$ and $\vec{v}_3$ are parallel. We will not discuss this case in detail, but it's good to be aware of it.

Some systems have the property that both solutions of the homogeneous equation decrease in magnitude and tend towards the origin as $t \to \infty$:

$$\lim_{t\to\infty} \vec{x}_1(t) = \lim_{t\to\infty} \vec{x}_2(t) = \vec{0}$$

For example, this happens in the real eigenvalue case if both eigenvalues are negative. It also happens in the complex eigenvalue case, if the complex eigenvalues have negative real part.

In either case, the limiting value of any solution of the inhomogeneous equation will be $\vec{x}_{eq}$:

$$\begin{aligned} \lim_{t\to\infty} \vec{x}(t) &= \lim_{t\to\infty} \left[\vec{x}_{eq} + c_1\vec{x}_1(t) + c_2\vec{x}_2(t)\right], \\ &= \lim_{t\to\infty} \vec{x}_{eq} + c_1 \lim_{t\to\infty} \vec{x}_1(t) + c_2 \lim_{t\to\infty} \vec{x}_2(t) \\ &= \vec{x}_{eq} \end{aligned}$$

In the context of first order autonomous *equations*, we had a word for an equilibrium point with this property: we said that an equilibrium point was **stable** if all solutions starting at nearby values were drawn back towards the equilibrium, and approached it in the limit as $t \to \infty$.

The same concept and terminology can be applied to systems of equations as well. For linear autonomous systems, we have the following stability criterion:

Stability Criterion for Linear Systems. A linear autonomous system

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$

has a stable equilibrium point $\vec{x}_{eq}$ if and only if every eigenvalue of A has a negative real part.

While we have only proved this criterion only for systems of two equations with distinct real eigenvalues and/or complex eigenvalues, it is true in general for linear autonomous systems with any number of equations.

There is also a generalization to *nonlinear* systems, which you will be guided through in the problem set.