Week 6

Statistics 251

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Where are we?

Change of variable

Joint distributions

Independent random variables

Change of variables (multivariate)

First question

Let X be uniformly distributed over $(0, 1)$, what is the distribution of $Y = X^n$?
Consider,

$$F_Y(y) = \mathbb{P}(Y \leq y)$$

So $f_Y(y) =$

More generally...

Let X be a continuous random variable with pdf f_X, what is the distribution of $Y = |X|$?
Consider,

$$F_Y(y) = \mathbb{P}(Y \leq y) \quad \text{where } y \geq 0$$

So $f_Y(y) =$

Theorem

Let X be a continuous random variable having probability density function f_X. Suppose that $g(x)$ is a strictly monotonic (increasing or decreasing), differentiable (and thus continuous) function of x. Then the random variable Y defined by $Y = g(X)$ has a probability density function given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

where $g^{-1}(y)$ is defined to equal that value of x such that $g(x) = y$.

Example: $Y = X^n$

Let X be a continuous nonnegative random variable with density function f, and let $Y = X^n$. Find f_Y, the probability density function of Y.

Example: Gamma Distribution

If X follows a gamma distribution with parameter (α, λ), then cX $(c > 0)$ follows a Gamma distribution with parameter $(\alpha, \lambda/c)$

Where are we?

Joint distribution for discrete r.v.

If X and Y assume values in $\{1, 2, ..., n\}$then we can view $A_{i,j} = \mathbb{P}\{X = i, Y = j\}$ as the entries of an $n \times n$ matrix.
What is $\mathbb{P}\{X = i\}$?

$\mathbb{P}\{X = i\} = \sum_{j=1}^{n} A_{i,j}$. What about $\mathbb{P}\{Y = j\}$?

Given the joint distribution of X and Y, we sometimes call distribution of X (ignoring Y) and distribution of Y (ignoring X) the marginal distributions.
In general, when X and Y are jointly defined discrete random variables, we write$\mathbb{P}(x, y) = \mathbb{P}_{X,Y}(x, y) = \mathbb{P}\{X = x, Y = y\}$.

Joint distribution for continuous r.v.

Similar to discrete version, if given random variables X and Y, define $F(a, b) = P\{X \leq a, Y \leq b\}$,

how to define marginal cumulative distribution function?
$F_X(a) =$

$F_Y(b) =$

Joint density functions for continuous r.v.

Given random variables X and Y, with $F(a, b) = P\{X \leq a, Y \leq b\}$. How to construction density function f?

Let $f(x, y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F(x, y)$.
We should first verify, for $A = \{(X, Y) : X \leq a, Y \leq b\}$,

$$\mathbb{P}\{(X, Y) \in A\} = \int_A f(x, y) dxdy$$

.

Then we should show it works for strips, rectangles and general open sets.

Example: From density to probability

The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute

$\mathbb{P}\{X > 1, Y < 1\}$

$\mathbb{P}\{X < Y\}$

$\mathbb{P}\{X < a\}$

Example: Density of functions of r.v.

The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} e^{-x}e^{-y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

What is the density function of random variable $\frac{X}{Y}$

Where are we?

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Definition

We say X and Y are independent if for any two (measurable) events A and B we have

$$\mathbb{P}\{X \in A, Y \in B\} = \mathbb{P}\{X \in A\}\mathbb{P}\{Y \in B\}.$$

What about density function if X and Y are continous?

Example: Waiting time

A couple (A, B) decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 P.M., find the probability that the first to arrive has to wait longer than 10 minutes.

Example: Buffon's needle problem

A table is ruled with equidistant parallel lines a distance D apart. A needle of length L, where $L \leq D$, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)?

Where are we?

Formula

Let X_1 and X_2 be jointly continuous random variables with joint probability density function f_{X_1,X_2}. Suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$

Assume that the functions g_1 and g_2 satisfy

1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2, with solutions given by, say, $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.
2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the 2×2 determinant, i.e.

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_2}{\partial x_1}\frac{\partial g_1}{\partial x_2} \neq 0$$

   at all points (x_1, x_2).

Then Y_1 and Y_2 are jointly continuous with joint density function given by

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}$$

where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

Formula for more variables

Let $X = (X_1, \cdots, X_n)$ be jointly continuous random variables with joint probability density function f_X. Suppose that $Y = g(X)$ Assume that the functions g satisfy

1. The equations $y = g(x)$ can be uniquely solved for x in terms of y , with solutions given by, say, $x = h(y)$.
2. The functions g have continuous partial derivatives at all points x and are such that the $n \times n$ determinant, i.e.

   $$J(x) \neq 0$$

   at all points x.

Then Y are jointly continuous with joint density function given by

$$f_Y(y) = f_X(x)|J(x)|^{-1}$$

where $x = h(y)$.

Example: 2D transform

Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2} .

Example: general linear transform

Let $X = (X_1, \cdots, X_n)$ be jointly continuous random variables with joint probability density function f_X. Suppose that $Y = AX$ where A is a $n \times n$ matrix with $|A| \neq 0$. Find the joint density function of Y in terms of f_X .

Example: 2D normal r.v. in the polar coodinate

Let (X, Y) denote a random point in the plane, and assume that the rectangular coordinates X and Y are independent standard normal random variables. What is the joint distrubtion of (r, θ), the polar coordinate representation of (x, y)?
To be more specific, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} y/x$.

Summation of i.i.d. exp varaibles

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed exponential random variables with rate λ. Let

$$Y_i = X_1 + \cdots + X_i \quad i = 1, \cdots, n$$

1. Find the joint density function of $Y_1, \cdots, Y_n$.
2. Use the first result to find the density of Y_n.