Lecture 17: Sums and conditional Distributions

Statistics 251

Yi Sun and Zhongjian Wang

Department of Statistics
The University of Chicago

Lecture Outline

Sums of independent r.v.

Conditional distribution

Where are we?

Sums of independent r.v.

Conditional distribution

Sum of two independent r.v.

As we discussed in previous lecture, given X, Y independent, continous r.v. with density function f_X, f_Y, the density function of $X + Y$ can be written as,

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a - y) f_Y(y) dy = \int_{-\infty}^{\infty} f_X(x) f_Y(a - x) dx$$

Sums of uniform distributions

If X_1, X_2 are independent identical uniform distributed on $(0, 1)$, what is the distribution of $X_1 + X_2$?

Continued..

If X_1, X_2, $\cdots$, are independent identical uniform distributed on $(0,1)$. What is the expectation of N where

$$N = \min\{n : X_1 + X_2 + \cdots + X_n > 1\}$$

Continued..

If X_1, X_2, $\cdots$, are independent identical uniform distributed on $(0,1)$. What is the expectation of N where

$$N = \min\{n : X_1 + X_2 + \cdots + X_n > 1\}$$

Let F_n denote cummulative distribution function of $X_1 + \cdots + X_n$. By Mathematical Induction, we first try to prove
$F_n(x) = x^n/n!, \quad 0 \leq x \leq 1.$

Continued..

If X_1, X_2, $\cdots$, are independent identical uniform distributed on $(0, 1)$. What is the expectation of N where

$$N = \min\{n : X_1 + X_2 + \cdots + X_n > 1\}$$

Let F_n denote cummulative distribution function of $X_1 + \cdots + X_n$. By Mathematical Induction, we first try to prove $F_n(x) = x^n/n!, \quad 0 \leq x \leq 1.$

So $\mathbb{P}\{N > n\} = F_n(1)$

Sums of Normal distribution

Recall density function of a normal distribution with parameters (μ, σ^2) is given by

Sums of Normal distribution

Recall density function of a normal distribution with parameters (μ, σ^2) is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

Proposition: If X_1, X_2, $\cdots$, X_n are independent random random variables with respective parameters (μ_1, σ_1^2), (μ_2, σ_2^2), $\cdots$, (μ_1, σ_1^2), then $X_1 + X_2 + \cdots + X_n$ is a normal random variable with parameters $(\mu_1 + \mu_2 + \cdots + \mu_n, \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)$.
The proof is left as part of the homework.

Sums of Gamma distribution

Recall density function of a gamma distribution with parameters (α, λ) is given by

Sums of Gamma distribution

Recall density function of a gamma distribution with parameters (α, λ) is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Proposition: If X and Y are independent gamma random variables with respective parameters (s, λ) and (t, λ), then $X + Y$ is a gamma random variable with parameters $(s + t, \lambda)$.
The proof is also left as part of the homework.

Where are we?

Sums of independent r.v.

Conditional distribution

Definition: Discrete Case

Recall that, for any two events E and F, the conditional probability of E given F is defined, provided that $\mathbb{P}(F) > 0$, by

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(EF)}{P(F)}$$

Definition: Discrete Case

Recall that, for any two events E and F, the conditional probability of E given F is defined, provided that $\mathbb{P}(F) > 0$, by

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(EF)}{P(F)}$$

Hence, if X and Y are discrete random variables, it is natural to define the conditional probability mass function of X given that $Y = y$, by

$$\begin{aligned}\mathbb{P}_{X|Y}(x|y) &= \mathbb{P}\{X = x | Y = y\} \\ &= \frac{\mathbb{P}\{X = x, Y = y\}}{\mathbb{P}\{Y = y\}} \\ &= \frac{p(x, y)}{p_Y(y)}\end{aligned}$$

for all values of y such that $p_Y(y) > 0$.

Definition: Continuous Case

If X and Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X given that $Y = y$ is defined, for all values of y such that $f_Y(y) > 0$, by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

Definition: Continuous Case

If X and Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X given that $Y = y$ is defined, for all values of y such that $f_Y(y) > 0$, by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

Recall if X, Y are independent, then $f(x, y) =$
Now what is $\mathbb{P}_{X|Y}(x|y)$?

Example: Computation

Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0 < x < q, 0 < y < q \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathbb{P}\{X > 1 | Y = y\}$.

Example: Bivariate Normal Distribution

One of the most important joint distributions is the bivariate normal distribution. We say that the random variables X, Y have a bivariate normal distribution if, for constants μ_x, μ_y, $\sigma_x > 0$, $\sigma_y > 0$, $-1 < \rho < 1$, their joint density function is given by,

$$
\begin{aligned}
f(x,y) = & \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\Big(- \frac{1}{2(1-\rho^2)} \\
& \cdot \big((\frac{x-\mu_x}{\sigma_x})^2 + (\frac{y-\mu_y}{\sigma_y})^2 - 2\rho \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \big) \Big)
\end{aligned}
$$

Example: Bivariate Normal Distribution

One of the most important joint distributions is the bivariate normal distribution. We say that the random variables X, Y have a bivariate normal distribution if, for constants μ_x, μ_y, $\sigma_x > 0$, $\sigma_y > 0$, $-1 < \rho < 1$, their joint density function is given by,

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\Big(- \frac{1}{2(1-\rho^2)} \cdot \big((\frac{x-\mu_x}{\sigma_x})^2 + (\frac{y-\mu_y}{\sigma_y})^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \big) \Big)$$

Proposition:

1. Given $Y = y$, the random variable X is nromally distributed with mean $\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y)$ and variance $\sigma_x^2(1-\rho^2)$
2. The marginal distribution of X is normal with mean μ_x and variance σ_x^2.