# Lecture 15: Weak Taylor Approximation

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**Abstract**

Introducing weak schemes based on Ito-Taylor expansion and the convergence theorem.

## 1 Weak Euler Scheme

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n, \tag{1.1}$$

with initial data $Y_0 = X_0$.

*Weak Approximation* is for approximating the measure (or moments) related to the Ito SDE solution $X(t)$. One could replace $\Delta W_n$ by a simple two-point distributed r.v $\Delta\tilde{W}_n$ with:

$$Prob(\Delta\tilde{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}.$$

To study weak convergence of approximation, introduce space $H^{(l)}$ for functions of $x$, $l \in (0,1) \cup (1,2) \cup (2,3)$. $H^{(l)}$ consists of $u(x)$ such that $\partial_x^s u$ is Hölder continous with exponent $l - [l]$, $[l]$ integral part of $l$, $s$ an integer $\leq l$. Hölder norm of a function $v(x)$ is:

$$\|v\| = \sup_{x \neq x'} \frac{|v(x) - v(x')|}{|x - x'|^{l-[l]}}.$$

The $H^{(l)}$ norm is:

$$\|u\|_l = \|\partial_x^{[l]} u\| + \sum_{s \leq l} \sup |u^{(s)}(x)|.$$

The convergence of Euler weak approximation is:

**Theorem 1.1** *Let $X(t)$ be Ito SDE solution over $[0,T]$, $a(x)$, $b(x) \in H^{(l)}$, and let $Y^\delta(t)$ be Euler approximation with time step $\delta$. For any function $g \in H^{(l+2)}$:*

$$|E(g(X(T))) - E(g(Y^\delta(T)))| \leq K\delta^{\chi(l)},$$

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$$\chi(l) = \begin{cases} l/2, \text{ if } l \in (0,1), \\ 1/(3-l), \text{ if } l \in (1,2), \\ 1, \text{ if } l \in (2,3), \end{cases} \tag{1.2}$$

*and $K$ independent of $l$.*

**Remark 1.1** *If the coefficients $a$ and $b$ are slightly more differentiable than twice, the weak convergence is first order. When $l = 1$, namely, coefficients are Lipschitz, weak convergence is order 0.5.*

## 1.1 Convergence of Weak Euler

Let $f = f(t,x)$ be a Hölder continuous function of exponent $l$ in $x \in R^1$, $l/2$ in $t \in [0,T]$, such functions form the Hölder space $H_T^{(l)}$. Let $Y^\delta(t)$ be the Euler approximate solution of Ito SDE solution $X(t)$ starting from same initial data $X_0 = Y_0$. The noise increment $\Delta\tilde{W}$ satisfies:

$$E(|\Delta\tilde{W}|^3) + |E(\Delta\tilde{W})^2 - \Delta| \leq K\Delta^2. \tag{1.3}$$

**Lemma 1.1** *Suppose drift and diffusion $a$ and $b$ are bounded, then for any $\eta \in (0,1)$, there is a positive constant $K_\eta$ such that:*

$$|E(f(s, Y^\delta(s)) - f(\tau_{n_s}, Y^\delta_{n_s})|A_{\tau_{n_s}})| \leq K_\eta \|f\|_T^{(l)} \delta^{\chi(l)}, \tag{1.4}$$

*$s \in [0,T]$, $l \in [\eta, 1) \cup (1,2) \cup (2,3)$, $\chi$ is defined in (1.2).*

Proof: let $w_\epsilon(x) = \frac{1}{\epsilon} w(\frac{x}{\epsilon})$, the mollifier, define:

$$f^{h,\epsilon} = h^{-1} \int_t^{t+h} \int f(\min(u,T), y) w_\epsilon(x-y) dy du,$$

then:

$$\begin{aligned} \sup_{t,x} |f(t,x) - f^{h,\epsilon}(t,x)| &\leq \|f\|_T^{(l)} (h^{\min(l/2,1)} + \epsilon^{\min(l,1)}), & (1.5) \\ \sup_{t,x} |\partial_x^i f^{h,\epsilon}(t,x)| &\leq K\|f\|_T^{(l)} \epsilon^{\min(l-i,0)}, & (1.6) \\ \sup_{t,x} |\partial_t f^{h,\epsilon}(t,x)| &\leq K\|f\|_T^{(l)} h^{\min(-1+l/2,0)}, & (1.7) \end{aligned}$$

$i = 1,2$, min with 1 in (1.5) is due to first differencing of left had side; integer derivatives in (1.6)-(1.7) reduce exponent by 1.

We replace $f$ by $f^{h,\epsilon}$ and estimate errors.

$$\begin{aligned}
&|E(f(s,Y^\delta(s)) - f(\tau_{n_s},Y^\delta_{n_s})|A_{\tau_{n_s}})| \\
&\le 2\sup_{t,x}|f(t,x) - f^{h,\epsilon}(t,x)| \\
&+|E(f^{h,\epsilon}(s,Y^\delta(s)) - f^{h,\epsilon}(\tau_{n_s},Y^\delta_{n_s})|A_{\tau_{n_s}})|
\end{aligned} \tag{1.8}$$

The second term of (1.8) is estimated by an approximate Ito formula thanks to (1.3) and (1.5)-(1.7), skipping superscript $\delta$ on $Y^\delta$:

$$\begin{aligned}
\le\ & |E(\int_{\tau_{n_s}}^s du\, \partial_t f^{h,\epsilon}(u,Y(u)) + \frac{1}{2}b(\tau_{n_s},Y_{n_s})f^{h,\epsilon}_{xx}(u,Y(u)) \\
& +a(\tau_{n_s},Y_{n_s})f^{h,\epsilon}_x(u,Y(u))|A_{\tau_{n_s}})| + K_1\delta^2 \\
& \le K\|f\|^{(l)}_T(h^{\min(-1+l/2,0)} + \epsilon^{\min(l-2,0)})\delta.
\end{aligned} \tag{1.9}$$

So:

$$\begin{aligned}
&|E(f(s,Y^\delta(s)) - f(\tau_{n_s},Y^\delta_{n_s})|A_{\tau_{n_s}})| \\
&\le K\|f\|^{(l)}_T[\inf_{h\in(0,1)}(h^{\min(l/2,1)} + h^{\min(-1+l/2,0)}\delta) \\
&+ \inf_{\epsilon\in(0,1)}(\epsilon^{\min(l,1)} + \epsilon^{\min(l-2,0)}\delta)], \\
&\le K_\eta\|f\|^{(l)}_T\delta^{\chi(l)},
\end{aligned} \tag{1.10}$$

proof is finished.

**Proof of Theorem 1.1** Let:

$$L_0 = \partial_t + a(x)\partial_x + \frac{1}{2}b(x)\partial_{xx},$$

there is unique solution of final value problem:

$$L_0 v = 0, \quad v(T,x) = g(x), \tag{1.11}$$

such that:

$$\|v\|^{(l+2)}_T \le K\|g\|^{(l+2)}, \tag{1.12}$$

and by Ito:

$$E(v(0,X_0)) = E(v(T,X_T)) = E(g(X_T)).$$

It follows by Ito formula and triangle inequality:

$$\begin{aligned}
& |E(g(X_T)) - E(g(Y(T)))| \\
= & |E(v(0,X_0)) - E(v(T,Y(T)))| = |E(v(T,Y(T))) - E(v(0,Y_0))| \\
= & |E(\int_0^T [\frac{1}{2} b(Y_{n_s}) v_{xx} + a(Y_{n_s}) v_x + v_t - L_0 v](s, Y(s)) ds)| + O(\delta) \\
\leq & \int_0^T |E([b(Y_{n_s}) - b(Y(s))] v_{xx}(s, Y(s)))| ds \\
& + \int_0^T |E([a(Y_{n_s}) - a(Y(s))] v_x(s, Y(s)))| ds + O(\delta) \\
\leq & \int_0^T |E(b(Y_{n_s}) v_{xx}(\tau_{n_s}, Y_{n_s}) - b(Y(s)) v_{xx}(s, Y(s)) | A_{\tau_{n_s}})| + O(\delta) \\
& + |E(b(Y_{n_s})[v_{xx}(\tau_{n_s}, Y_{n_s}) - v_{xx}(s, Y(s))] | A_{\tau_{n_s}})| \, ds \\
+ & \ldots\ldots
\end{aligned}$$

.... refer to similar terms on drift. Note that $bv_{xx}$, $v_{xx}$, $av_x$, $v_x$ all belong to $H_T^{(l)}$ due to (1.12). Applying the lemma, we prove the weak convergence theorem of the Euler method.

# 2 Higher Order Weak Schemes

## 2.1 Order 2 Weak Schemes

Adding all of the double stochastic integrals from Ito-Taylor expansions gives the order 2 weak scheme:

$$\begin{aligned}
Y_{n+1} = & Y_n + a\Delta + b\Delta W + \frac{1}{2} bb'((\Delta W)^2 - \Delta) \\
& + a'b\Delta Z + \frac{1}{2}(aa' + \frac{1}{2} a'' b^2)\Delta^2 \\
& + (ab' + \frac{1}{2} b'' b^2)(\Delta W \Delta - \Delta Z)
\end{aligned} \tag{2.13}$$

$\Delta Z = \int_0^\Delta W_s \, ds$. Here $\Delta W$ and $\Delta Z$ are generated jointly by mapping independent unit Gaussians $U_i$, $i = 1, 2$.

$$\Delta W = U_1 \sqrt{\Delta}, \ \Delta Z = \frac{1}{2} \Delta^{3/2} (U_1 + \frac{1}{\sqrt{3}} U_2).$$

Simplified weak schemes are constructed by replacing $\Delta W$ by a similarly distributed

$\Delta\hat{W}$, and $\Delta Z$ by $\frac{1}{2}\Delta\hat{W}\Delta$ to approximate $E(\Delta Z\Delta W) = \Delta^2/2$:

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta\hat{W} + \frac{1}{2}bb'((\Delta\hat{W})^2 - \Delta) \\ &+\frac{1}{2}(a'b + ab' + \frac{1}{2}b''b^2)\Delta\hat{W}\Delta \\ &+\frac{1}{2}(aa' + \frac{1}{2}a''b^2)\Delta^2, \end{aligned} \tag{2.14}$$

where $\Delta\hat{W}$ satisfies the moment condition:

$$E(|\Delta\hat{W}|^5) + |E((\Delta\hat{W})^2) - \Delta| + |E((\Delta\hat{W})^4) - 3\Delta^2| \leq K\Delta^3, \tag{2.15}$$

One may choose $\hat{W}$ as $N(0, \Delta)$, or 3-point random variable taking $\pm\sqrt{3\Delta}$ with prob 1/6 each, and zero with prob 2/3.

**General Multi-dimensional case** In the general multi-dimensional case $d, m = 1, 2, \ldots$ the $k$ th component of the order 2.0 weak Taylor scheme takes the form

$$\begin{aligned} Y_{n+1}^k =& Y_n^k + a^k\Delta + \frac{1}{2}L^0a^k\Delta^2 \\ &+ \sum_{j=1}^{m}\left\{b^{k,j}\Delta W^j + L^0b^{k,j}I_{(0,j)} + L^ja^kI_{(j,0)}\right\} \\ &+ \sum_{j_1,j_2=1}^{m} L^{j_1}b^{k,j_2}I_{(j_1,j_2)} \end{aligned} \tag{2.16}$$

For weak convergence we can substitute simpler random variables the multiple Ito integrals. In this way we obtain from (2.16) the following simplified order 2.0 weak Taylor scheme with $k$ th component

$$\begin{aligned} Y_{n+1}^k =& Y_n^k + a^k\Delta + \frac{1}{2}L^0a^k\Delta^2 \\ &+ \sum_{j=1}^{m}\left\{b^{k,j} + \frac{1}{2}\Delta\left(L^0b^{k,j} + L^ja^k\right)\right\}\Delta\hat{W}^j \\ &+ \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1}b^{k,j_2}\left(\Delta\hat{W}^{j_1}\Delta\hat{W}^{j_2} + V_{j_1,j_2}\right) \end{aligned}$$

Here the $\Delta\hat{W}^j$ for $j = 1, 2, \ldots, m$ are independent random variables satisfying (2.15) and the $V_{j_1,j_2}$ are independent two-point distributed random variables with

$$P\left(V_{j_1,j_2} = \pm\Delta\right) = \frac{1}{2}$$

for $j_2 = 1, \ldots, j_1 - 1$,

$$V_{j_1,j_1} = -\Delta$$

and

$$V_{j_1,j_2} = -V_{j_2,j_1}$$

## 2.2 Order 3 Schemes

Consider $d = m = 1$,

$$\begin{aligned} Y_{n+1} = Y_n + a\Delta + b\Delta W + L^0 a I_{(0,0)} + L^1 a I_{(1,0)} + L^0 b I_{(0,1)} + L^1 b I_{(1,1)} \\ + L^0 L^0 a I_{(0,0,0)} + L^0 L^1 a I_{(0,1,0)} + L^1 L^0 a I_{(1,0,0)} + L^1 L^1 a I_{(1,1,0)} \\ + L^0 L^0 b I_{(0,0,1)} + L^0 L^1 b I_{(0,1,1)} + L^1 L^0 b I_{(1,0,1)} + L^1 L^1 b I_{(1,1,1)} \end{aligned}$$

By comparing moments, we propose,

$$\begin{aligned} Y_{n+1} = Y_n &+ a\Delta + b\Delta\tilde{W} + \frac{1}{2} L^1 b \left\{ (\Delta\tilde{W})^2 - \Delta \right\} \\ &+ L^1 a \Delta\tilde{Z} + \frac{1}{2} L^0 a \Delta^2 + L^0 b \{\Delta\tilde{W}\Delta - \Delta\tilde{Z}\} \\ &+ \frac{1}{6} \left( L^0 L^0 b + L^0 L^1 a + L^1 L^0 a \right) \Delta\tilde{W} \Delta^2 \\ &+ \frac{1}{6} \left( L^1 L^1 a + L^1 L^0 b + L^0 L^1 b \right) \left\{ (\Delta\tilde{W})^2 - \Delta \right\} \Delta \\ &+ \frac{1}{6} L^0 L^0 a \Delta^3 + \frac{1}{6} L^1 L^1 b \left\{ (\Delta\tilde{W})^2 - 3\Delta \right\} \Delta\tilde{W} \end{aligned}$$

where $\Delta\tilde{W}$ and $\Delta\tilde{Z}$ are correlated Gaussian random variables with

$$\Delta\tilde{W} \sim N(0; \Delta), \quad \Delta\tilde{Z} \sim N\left(0; \frac{1}{3}\Delta^3\right)$$

and covariance

$$E(\Delta\tilde{W}\Delta\tilde{Z}) = \frac{1}{2}\Delta^2.$$

# 3 General Rule and Convergence

In general, a weak order $\beta = 1,\ 2,\ 3, \cdots$ scheme needs all of the multiple Ito integrals from the Ito-Taylor expansion in the set $\Gamma_\beta = \{\alpha : \ l(\alpha) \leq \beta\}$. Here $l$ is the length of the index $\alpha$. Note that is different from the strong scheme index set $A_\gamma$ which also depends on the number of zeros in the index $n(\alpha)$.

**Theorem 3.1** *Let $Y^\delta$ be a time discrete approximation of an autonomous Ito process $X$ corresponding to a time discretization $(\tau)_\delta$, such that all moments of the initial value $X_0$ exist, that is*

$$E\left(|X_0|^i\right) < \infty$$

*for $i = 1, 2, \ldots$, and such that $Y_0^\delta$ converges weakly with order $\beta$ to $X_0$ as $\delta \to 0$ for some fixed $\beta = 1.0, 2.0, \ldots$. Assume that $a(x)$, $b(x)$ are $C^{2(\beta+1)}$ and all derivatives up to $2(\beta+1)$ have polynomial growth in large $x$. In addition, suppose that for each $p = 1, 2, \ldots$ there exist constants $K < \infty$ and $r \in \{1, 2, \ldots\}$, which do not depend on $\delta$, such that for each $q \in \{1, \ldots, p\}$*

$$E\left(\max_{0\le n\le n_T} \left|Y_n^\delta\right|^{2q} \mid \mathcal{A}_0\right) \le K\left(1 + \left|Y_0^\delta\right|^{2r}\right)$$

*and $E\left(\left|Y_{n+1}^\delta - Y_n^\delta\right|^{2q} \mid \mathcal{A}_{\tau_n}\right) \le K\left(1 + \max_{0\le k\le n}\left|Y_k^\delta\right|^{2r}\right)(\tau_{n+1} - \tau_n)^q$ for $n = 0, 1, \ldots, n_T - 1$, and such that*

$$\left|E\left(\prod_{h=1}^{l}\left(Y_{n+1}^{\delta,p_h} - Y_n^{\delta,p_h}\right) - \prod_{h=1}^{l}\left(\sum_{\alpha\in\Gamma_\beta\setminus\{v\}} f_\alpha^{p_h}\left(\tau_n, Y_n^\delta\right) I_{\alpha,\tau_n,\tau_{n+1}}\right) \mid \mathcal{A}_{\tau_n}\right)\right| \le K\left(1 + \max_{0\le k\le n_T}\left|Y_k^\delta\right|^{2r}\right)\delta^\beta\left(\tau_{n+1} - \tau_n\right) \qquad (3.17)$$

*for all $n = 0, 1, \ldots, n_T - 1$ and $(p_1, \ldots, p_l) \in \{1, \ldots, d\}^l$, where $l = 1, \ldots 2\beta + 1$ and $Y^{\delta,p_h}$ denotes the $p_h$ th component of $Y^\delta$. Then the time discrete approximation $Y^\delta$ converges weakly with order $\beta$ as $\delta \to 0$ to the Ito process $X$ at time $T$.*

A straight forward corollary follows,

**Corollary 3.1** *Let $X(t)$ be an autonomous Ito SDE solution over $[0, T]$. Let $Y^\delta$ be solution of a weak scheme of order $\beta = 1, 2, 3, \cdots$, with exact Brownian increment. Then for any function $g \in C^{2(\beta+1)}$ whose derivatives up to $2(\beta+1)$ have polynomial growth in large $x$,*

$$|E(g(X(T))) - E(g(Y^\delta(T)))| \le K_g \delta^\beta,$$

*$K_g$ independent of $\delta$.*

Note, left hand side of (3.17) is zero with exact Brownian increment.