# Lecture 5: Strong and Weak Solution

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**Abstract**

Strong and Weak Solution of SDE

# 1 Integral Formulation of SDE

SDE:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{1.1}$$

and SIE:

$$X_t = X_0 + \int_0^t a(t, X_s)ds + \int_0^t b(s, X_s)dW_s. \tag{1.2}$$

*Strong solution:* for each BM $W_t$ and its filtration $A_t$, each initial data $X_0$, there is a process $X_t$, $t \geq 0$, with continuous sample path such that $X_t$ is adapted to $A_t$, and a solution of SIE (1.2).

*Uniqueness:* for given initial data $X_0$, there is only one solution to SIE (1.2) either in the mean square sense or pathwise sense: $P(\sup_{t\in[0,T]} |X_t - \tilde{X}_t| > 0) = 0$.

## 1.1 Sufficient Conditions

(A1) $a$, $b$ are measurable in $(t, x)$, Lipschitz in $x$:

$$\begin{aligned} |a(t,x) - a(t,y)| &\leq K|x-y|, \\ |b(t,x) - b(t,y)| &\leq K|x-y|, \end{aligned} \tag{1.3}$$

for any $t \in [0,T]$, $x$, $y$.

(A2) Linear growth bound:

$$|a(t,x)|^2 \leq K^2(1+|x|^2),\ |b(t,x)|^2 \leq K^2(1+|x|^2), \tag{1.4}$$

for any $t \in [0,T]$, $x$, $y$.

(A3) $X_0$ is $A_0$ measurable, $E(|X_0|^2) < \infty$.

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## 1.2 Gronwall Inequality

If $\alpha(t) \geq 0$ satisfies:

$$\alpha(t) \leq \beta(t) + L \int_0^t \alpha(s)\, ds,$$

then:

$$\alpha(t) \leq \beta(t) + L \int_0^t e^{L(t-s)} \beta(s)\, ds,$$

for $t \in [0, T]$.

## 1.3 Uniqueness

### 1.3.1 Non-rigorous derivation

Suppose $X_t$, $Y_t$ are two solutions of SIE with same initial data, $Z_t = X_t - Y_t$:

$$\begin{aligned} Z_t &= \int_0^t (a(s, X_s) - a(s, Y_s)) ds \\ &\quad + \int_0^t (b(s, X_s) - b(s, Y_s)) dW_s. \end{aligned} \tag{1.5}$$

By Cauchy-Schwarz, mean square property of Ito integral, (A1):

$$\begin{aligned} E(|Z_t|^2) &\leq 2E[|\int_0^t (a(s, X_s) - a(s, Y_s)) ds|^2] \\ &+ 2E[|\int_0^t (b(s, X_s) - b(s, Y_s)) dW_s|^2] \\ &\leq 2T \int_0^t E[|a(s, X_s) - a(s, Y_s)|^2] ds \\ &\quad +2 \int_0^t E[|b(s, X_s) - b(s, Y_s)|^2] ds \\ &\leq L \int_0^t E[|Z_s|^2] ds, \end{aligned} \tag{1.6}$$

$L = 2(T + 1)K^2$. Gronwall implies $E(|Z_t|^2) = 0$, uniqueness in the mean square sense. With more work, one can show that pathwise uniqueness also holds.

### 1.3.2 Rigorous Version

Let $X_t$ and $\tilde{X}_t$ be two such solutions of SIE on $[0, T]$ with, almost surely, continuous sample paths. **Since they may not have finite second moments**, we shall use the following truncation procedure: for $N > 0$ and $t \in [0, T]$ we define

$$I_t^{(N)}(\omega) = \begin{cases} 1 & : \ |X_u(\omega)|, \left|\tilde{X}_u(\omega)\right| \leq N \text{ for } 0 \leq u \leq t \\ 0 & : \ \text{otherwise} \end{cases}$$

Obviously $I_t^{(N)}$ is $\mathcal{A}_t$ -measurable and $I_t^{(N)} = I_t^{(N)} I_s^{(N)}$ for $0 \le s \le t$. Consequently the integrals in the following expression are meaningful:

$$Z_t^{(N)} = I_t^{(N)} \int_0^t I_s^{(N)} \left( a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right) \right) ds \tag{1.7}$$

$$+ I_t^{(N)} \int_0^t I_s^{(N)} \left( b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right) \right) dW_s$$

where $Z_t^{(N)} = I_t^{(N)} \left( X_t - \tilde{X}_t \right)$. From the Lipschitz condition we then have

$$\max \left\{ \left| I_s^{(N)} \left( a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right) \right) \right|, \left| I_s^{(N)} \left( b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right) \right) \right| \right\}$$
$$\le K I_s^{(N)} \left| X_s - \tilde{X}_s \right| \le 2KN \tag{1.8}$$

for $0 \le s \le t$. Thus the second order moments exist for $Z_t^{(N)}$ and the two integrals in (1.7). Using the inequality $(a+b)^2 \le 2\left(a^2 + b^2\right)$, the Cauchy-Schwarz inequality, we obtain from

$$\begin{aligned} E\left( \left| Z_t^{(N)} \right|^2 \right) \le & 2E\left( \left| \int_0^t I_s^{(N)} \left( a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right) \right) ds \right|^2 \right) \\ & + 2E\left( \left| \int_0^t I_s^{(N)} \left( b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right) dW_s \right|^2 \right) \\ \le & 2T \int_{t_0}^t E\left( \left| I_s^{(N)} \left( a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right) \right) \right|^2 \right) ds \\ & + 2 \int_0^t E\left( \left| I_s^{(N)} \left( b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right) \right) \right|^2 \right) ds \end{aligned}$$

which we combine with (1.8) to get

$$E\left( \left| Z_t^{(N)} \right|^2 \right) \le L \int_0^t E\left( \left| Z_s^{(N)} \right|^2 \right) ds$$

for $t \in [0, T]$ where $L = 2\left(T+1\right) K^2$. We then apply the Gronwall inequality to conclude that

$$E\left( \left| Z_t^{(N)} \right|^2 \right) = E\left( \left| I_t^{(N)} \left( X_t - \tilde{X}_t \right) \right|^2 \right) = 0$$

and hence that $I_t^{(N)} X_t = I_t^{(N)} \tilde{X}_t$, w.p.1, for each $t \in [0, T]$. Note,

$$P\left( I_t^{(N)} \not\equiv 1 \forall t \in [0, T] \right) \le P\left( \sup_{0 \le t \le T} \left| X_t \right| > N \right) + P\left( \sup_{0 \le t \le T} \left| \tilde{X}_t \right| > N \right).$$

Since the sample paths are continuous almost surely they are bounded almost surely, we can make the probability on the right arbitrarily small by taking $N$ sufficiently large. This means that $P\left(X_t \neq \tilde{X}_t\right) = 0$ for each $t \in [0,T]$.

And then $P\left(X_t \neq \tilde{X}_t : t \in D\right) = 0$ for any **countably** dense subset $D$ of $[0,T]$. As the solutions are continuous and coincide on a countably dense subset of $[0,T]$, they must coincide, almost surely, on the entire interval $[0,T]$.

## 1.4 Existence

Picard iteration:

$$X_t^{n+1} = X_0 + \int_0^t a(s, X_s^n)ds + \int_0^t b(s, X_s^n)dW_s, \tag{1.9}$$

$X_t^n$ all measurable in $A_t$, $X_t^0 = X_0$. Second moment estimate:

$$\begin{aligned} E[|X_t^{n+1}|^2] &\leq 3E[|X_0|^2] + 3E[|\int_0^t a(s, X_s^n)ds|^2] \\ &+ 3E[|\int_0^t b(s, W_s)dW_s|^2] \\ &\leq 3E[|X_0|^2] + 3TE[\int_0^t |a(s, X_s^n)|^2 ds] \\ &\quad +3E[\int_0^t |b(s, X_s^n)|^2 ds] \\ &\leq 3E[|X_0|^2] + 3(T+1)K^2 E[\int_0^t 1 + |X_s^n|^2] \end{aligned} \tag{1.10}$$

implying $\sup_{t\in[0,T]} E[|X_t^n|^2] \leq C_0 < \infty$, for all $n$.
As in uniqueness:

$$E(|X_t^{n+1} - X_t^n|^2) \leq L \int_0^t E(|X_s^n - X_s^{n-1}|^2)ds \tag{1.11}$$

iterating:

$$E(|X_t^{n+1} - X_t^n|^2) \leq \frac{L^n}{(n-1)!} \int_0^t (t-s)^{n-1} E[|X_s^1 - X_s^0|^2]ds. \tag{1.12}$$

By growth bound (A2):

$$E[|X_t^1 - X_t^0|^2] \leq L \int_0^t (1 + E[|X_s^0|^2]) \leq C_1 < \infty, \tag{1.13}$$

then (1.12):

$$E(|X_t^{n+1} - X_t^n|^2) \leq C_1 L^n t^n / n!,$$

or:

$$\sup_{t\in[0,T]} E(|X_t^{n+1} - X_t^n|^2) \leq C_1 L^n T^n / n!,$$

implying $X_t^n$ converges in mean square sense for $t \in [0, T]$.

Note: $\mathcal{L}_T^2$ equipped with,

$$\|f\|_{2,T} = \sqrt{\int_0^T E\left(f(t,\cdot)^2\right) dt} \tag{1.14}$$

is a Banach space space.

The limit is a solution of SIE (some work needed to achieve path-wise convergence and then pass the limit).

## 1.5 Generalized results

### 1.5.1 Lipschitz Condition

Lipschitz condition in $b$ can be replaced by Yamada condition:

$$|b(t,x) - b(t,y)| \leq \rho(|x-y|),\ \rho(0) = 0,\ \int_{0^+} \rho^{-2}(u)du = +\infty. \tag{1.15}$$

One can take $\rho(u) = u^q$, $q \in [1/2, 1]$.
*Example:*

SDE $dX_t = |X_t|^q dW_t$, $X_0 = 0$, has unique solution if $q \in [1/2, 1]$.

But SDE:

$$dX_t = \frac{1}{3} X_t^{1/3} dt + X_t^{2/3} dW_t, \tag{1.16}$$

with initial data $X_0 = 0$ has nontrivial solution (hence nonuniqueness):

$$X_t = (W_t/3)^3.$$

### 1.5.2 Growth Condition

Growth condition on $a$ can be replaced by:

$$xa(t,x) \leq K(1 + |x|^2),$$

allowing $a = x - x^3$.

But SDE:

$$dY_t = -\frac{1}{2} e^{-2Y_t} dt + e^{-Y_t} dW_t, \tag{1.17}$$

has unique solution:

$$Y_t = \ln(W_t + e^{Y_0}),$$

valid until time:

$$T = T(Y_0(\omega)) = \min\{t \geq 0 : W_t(\omega) + e^{Y_0(\omega)} = 0.\}$$

Growth condition (A2) is not satisfied.
Higher moments: if $E[|X_0|^{2n}] < \infty$,

$$E[|X_t|^{2n}] \leq (1 + E[|X_0|^{2n}])e^{Ct}. \tag{1.18}$$

## 1.6 Stability

If as $\epsilon \to 0$:

$$E[|X_0^\epsilon - X_0|^2] \to 0,$$

$$\sup_{|x|\leq N} |a^\epsilon(t,x) - a(t,x)| \to 0,\ t \in [0,T],\ \forall N,$$

$$\sup_{|x|\leq N} |b^\epsilon(t,x) - b(t,x)| \to 0,\ t \in [0,T],\ \forall N,$$

then (by Gronwall Inequality):

$$\sup_{t\in[0,T]} E[|X_t^\epsilon - X_t|^2] \to 0.$$

*Example:* Solutions of SDE:

$$dX_t^\epsilon = a(t, X_t^\epsilon)dt + \epsilon dW_t, \tag{1.19}$$

converge in the mean square sense to those of deterministic ODE: $X_t' = a(t, X_t)$ with same initial data.

# 2 Diffusion Process and Weak Solution

## 2.1 Strong Solution as a diffusion process

Recall definition of Diffusion process,
Markov process with transition density is called *diffusion process* if the following limits exist:
Jump:

$$\lim_{t \to s^+} \frac{1}{t-s} \int_{|y-x|>\epsilon} p(s,x;t,y)dy = 0,$$

Drift:

$$\lim_{t \to s^+} \frac{1}{t-s} \int_{|y-x|\le\epsilon} (y-x)p(s,x;t,y)dy = a(s,x),$$

Diffusion:

$$\lim_{t \to s^+} \frac{1}{t-s} \int_{|y-x|\le\epsilon} (y-x)^2\, p(s,x;t,y)dy = b^2(s,x).$$

*Theorem* Assume that $a$ and $b$ are continuous and that A1-A3 hold. Then the solution $X_t$ of (1.1) for any fixed initial value $X_{t_0}$ is a diffusion process on $[t_0, T]$ with drift $a(t,x)$ and diffusion coefficient $b(t,x)$.

## 2.2 Diffusion process as a Weak Solution

Note as we turn investigate strong solution as a diffusion process, we only verify the transition density of the distribution.

Given a diffusion process, $Y$, on $[0,T]$ with drift $a(t,y)$ and strictly positive diffusion coefficient $b(t,y)$. Under some assumptions (see Theorem 4.7.1 in K-L's book), we define functions $g$ and $\bar{a}$ by

$$g(t,y) = \int_0^y \frac{dx}{b(t,x)} \tag{2.20}$$

and

$$\bar{a}(t,z) = \left(\frac{\partial g}{\partial t} + a\frac{\partial g}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 g}{\partial y^2}\right)\left(t, g^{-1}(t,z)\right) \tag{2.21}$$

with $a$ and $b$ evaluated at $(t,y)$, where $y = g^{-1}(t,z)$ is the inverse of $z = g(t,y)$ Then we define a process $Z_t = g\left(t, Y_t\right)$, which is a diffusion process with drift $\bar{a}(t,z)$ and diffusion coefficient 1.

And the process

$$\tilde{W}_t = Z_t - Z_0 - \int_0^t \bar{a}\left(s, Z_s\right) ds \tag{2.22}$$

which will turn out to be a Wiener process. (Some proof needed here.)

Consequently (2.22) will be equivalent to the stochastic differential equation

$$dZ_t = \bar{a}\,(t, Z_t)\,dt + 1d\tilde{W}_t$$

which, by (2.20),(2.21) and Ito's formula, will imply that $Y_t$ is a solution of the stochastic differential equation

$$dY_t = a\,(t, Y_t)\,dt + b\,(t, Y_t)\,d\tilde{W}_t.$$

# 3 Backward and Forward Representations

Let $X(t)$ be a diffusion process (solution of SDE) with drift $a(t,x)$, diffusion $b(t,x)$:

$$dX_t = adt + bdW_t,$$

consider the conditional expectation ($s < t$):

$$E(f(X_t)|X_s = x) = \int f(y)\,p(s,x;t,y)\,dy, \tag{3.23}$$

where $p(s,x;t,y)$ is the transition probability density function from $(s,x)$ to $(t,y)$. As a function of $(s,x)$, $p$ satisfies the *backward equation*:

$$p_s + \frac{1}{2}b^2(s,x)p_{xx} + a(s,x)p_x = 0. \tag{3.24}$$

Hence $u(s,x) = E(f(X_t)|X_s = x)$ solves (3.24) with final condition $u(t,x) = f(x)$.
**For the forward representation**, consider the *Autonomous case*, $a = a(x)$, $b = b(x)$. Then $p(s,t;x,y) = p(t-s;x,y)$, $p_s = -p_t$,

$$p_t = \frac{1}{2}b^2(x)p_{xx} + a(x)p_x,\ t > s, \tag{3.25}$$

$p(t;x,y) \to \delta(y-x)$, as $t \to 0+$. The transition probability density becomes fundamental solution of parabolic equation (3.25). As a function of $(t,x)$,

$$v(t,x) = E(f(X_t)|X_s = x), \tag{3.26}$$

solves:

$$v_t = \frac{1}{2}b^2(x)v_{xx} + a(x)v_x, \tag{3.27}$$

with initial data $v(s,x) = f(x)$.
Eq. (3.26) is a probabilistic representation formula of PDE (3.27). It can be generalized to include a lower order (potential) term as in Eqn:

$$w_t = \frac{1}{2}b^2(x)w_{xx} + a(x)w_x + V(x)w,\ t > 0, \tag{3.28}$$

initial data: $w(0,x) = f(x)$. The Feynmann-Kac formula is:

$$w(t,x) = E\left[\exp\{\int_0^t V(X(\tau))\, d\tau\} f(X(\tau))\right], \tag{3.29}$$

If the diffusion $b(x) \equiv 0$, F-K formula reduces to a solution formula of first order hyperbolic eqn by the method of characteristics.
**To derive (3.29)**, let:

$$T_t f = E\left[\exp\{\int_0^t V(X(\tau))\, d\tau\} f(X(\tau))\right],$$

a linear bounded (nonnegative) operator on the space of bounded continuous functions. Note:

$$\exp\{\int_0^t V(X_s)ds\} = 1 + \int_0^t V(X_s)ds + o(t),$$

as $t \to 0+$. We have for any $f(x)$ in the domain of $T_t$:

$$\begin{aligned} \frac{T_t f(x) - f(x)}{t} &= \frac{1}{t}\left(E[f(X_t)e^{\int_0^t V(X_s)ds}] - f(x)\right) \\ &= \frac{1}{t}(E[f(X_t)] - f(x)) + \frac{1}{t}E[f(X_t)\int_0^t V(X_s)ds] \\ &\to (b^2(x)f_{xx}/2 + a(x)f_x) + V(x)f. \end{aligned} \tag{3.30}$$

We have used (3.25) for the limit of first term.
**To generalize F-K to nonautonomous case,** treat $t$ as a parameter,

$$\begin{aligned} dX_s^{t,x} &= a(t_s^{t,x}, X_s^{t,x})ds + b(t_s^{t,x}, X_s^{t,x})dW_s, \\ dt_s^{t,x} &= -ds, \end{aligned} \tag{3.31}$$

$X_0^{t,x} = x$, $t_0^{t,x} = t$, symmetrically extending $a$, $b$: $a(-\tau, x) = a(\tau, x)$ etc. *View (3.31) as a diffusion process on $(t,x) \in R^2$ with time $s$.* Eqs (3.31) are autonomous, and define a Markov process $(t_s^{t,x}, X_s^{t,x}, P)$. We then apply F-K (3.29). The result is:

$$w(t,x) = Ef(X_t^{t,x})\exp\{\left[\int_0^t V(t-s, X_s^{t,x})ds\right]\}, \tag{3.32}$$

solves eqn:

$$w_t = \frac{1}{2}b^2(t,x)w_{xx} + a(t,x)w_x + V(t,x)w, \tag{3.33}$$

$w(0,x) = f(x)$.

All results generalize to higher space dimensions.