Lecture 5: Strong and Weak Solution

Zhongjian Wang*

Abstract

Strong and Weak Solution of SDE

1 Integral Formulation of SDE

Recall definition of SDE:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{1.1}$$

and the corresponding SIE:

$$X_t = X_0 + \int_0^t a(t, X_s)ds + \int_0^t b(s, X_s)dW_s. \tag{1.2}$$

Strong solution is defined for each BM W_t and its filtration A_t, each initial data X_0, there is a process X_t, $t \geq 0$, with continuous sample path such that X_t is adapted to A_t, and a solution of SIE (1.2).

Uniqueness of strong solution: for given initial data X_0, there is only one solution to SIE (1.2)

- in the mean square sense $E \inf |X_t - \tilde{X}_t|^2 dt = 0$;
- in the pathwise sense: $P(\sup_{t\in[0,T]} |X_t - \tilde{X}_t| > 0) = 0$.

Eg. Recall in last lecture when we define vector SDE, we need vector BM W_t whose entries W^1, W^2. W^1, W^2 share same joint distribution but they are not identical in MS sense or pathwise sense.

1.1 Sufficient Conditions

Throughout this course, we are expecting only to solve SDE with unique solution. Here we list three usually assumes condition that sufficiently provides existence and uniqueness.

*Department of Statistics, University of Chicago

(A1) Lipschitz: a, b are measurable in (t, x), Lipschitz in x:

$$\begin{aligned} |a(t,x) - a(t,y)| &\le K|x-y|, \\ |b(t,x) - b(t,y)| &\le K|x-y|, \end{aligned} \tag{1.3}$$

for any $t \in [0, T]$, x, y.

(A2) Linear growth bound:

$$|a(t,x)|^2 \le K^2(1+|x|^2), \; |b(t,x)|^2 \le K^2(1+|x|^2), \tag{1.4}$$

for any $t \in [0, T]$, x, y.

(A3) X_0 is A_0 measurable, $E(|X_0|^2) < \infty$.

1.2 Gronwall Inequality

Differential Form: Given interval $I = [a, b]$ on real line, if

$$u'(t) \le \beta(t)u(t), \quad t \in I^\circ$$

then u is bounded by :

$$u(t) \le u(a) \exp\left(\int_a^t \beta(s) \mathrm{d}s\right)$$

for all $t \in I$.
Integral Form (only works for deterministic integral): If $u(t)$ satisfies:
- (a) If β is non-negative and if u satisfies the integral inequality

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)\mathrm{d}s, \quad \forall t \in I,$$

then

$$u(t) \le \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)\mathrm{d}r\right) \mathrm{d}s, \quad t \in I.$$

- (b) If, in addition, the function α is non-decreasing, then

$$u(t) \le \alpha(t) \exp\left(\int_a^t \beta(s)\mathrm{d}s\right), \quad t \in I.$$

-(c) If, in addition, α, β are constant, then

$$u(t) \le \alpha \exp\Big(\beta(t-a)\Big)$$

1.3 Uniqueness

Suppose X_t, Y_t are two solutions of SIE with same initial data ($X_0 = Y_0$), $Z_t = X_t - Y_t$:

$$
\begin{aligned}
Z_t &= \int_0^t (a(s, X_s) - a(s, Y_s))ds \\
&\quad + \int_0^t (b(s, X_s) - b(s, Y_s))dW_s.
\end{aligned} \tag{1.5}
$$

By Cauchy-Schwarz, mean square property of Ito integral, (A1):

$$
\begin{aligned}
E(|Z_t|^2) &\leq 2E[|\int_0^t (a(s, X_s) - a(s, Y_s))ds|^2] \\
&+ 2E[|\int_0^t (b(s, X_s) - b(s, Y_s))dW_s|^2] \\
&\leq 2T \int_0^t E[|a(s, X_s) - a(s, Y_s)|^2]ds \\
&\quad +2 \int_0^t E[|b(s, X_s) - b(s, Y_s)|^2]ds \\
&\leq L \int_0^t E[|Z_s|^2]ds,
\end{aligned} \tag{1.6}
$$

$L = 2(T + 1)K^2$. Gronwall implies $E(|Z_t|^2) = 0$, uniqueness in the mean square sense. With more work, one can show that pathwise uniqueness also holds.

1.3.1 If $E|X_0|^2$ is unbounded

Let X_t and $\tilde{X}_t$ be two such solutions of SIE on $[0, T]$ with, almost surely, continuous sample paths. **Since they may not have finite second moments**, we shall use the following truncation procedure: for $N > 0$ and $t \in [0, T]$ we define

$$
I_t^{(N)}(\omega) = \begin{cases} 1 & : \quad |X_u(\omega)|, \left|\tilde{X}_u(\omega)\right| \leq N \text{ for } 0 \leq u \leq t \\ 0 & : \quad \text{otherwise} \end{cases}
$$

Obviously $I_t^{(N)}$ is $\mathcal{A}_t$ -measurable and $I_t^{(N)} = I_t^{(N)} I_s^{(N)}$ for $0 \leq s \leq t$. Consequently the integrals in the following expression are meaningful:

$$
Z_t^{(N)} = I_t^{(N)} \int_0^t I_s^{(N)} \left(a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right)\right) ds \tag{1.7}
$$

$$
+I_t^{(N)} \int_0^t I_s^{(N)} \left(b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right)\right) dW_s
$$

where $Z_t^{(N)} = I_t^{(N)}\left(X_t - \tilde{X}_t\right)$. From the Lipschitz condition we then have

$$\max\left\{\left|I_s^{(N)}\left(a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right)\right)\right|, \left|I_s^{(N)}\left(b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right)\right)\right|\right\} \\ \leq K I_s^{(N)}\left|X_s - \tilde{X}_s\right| \leq 2KN \tag{1.8}$$

for $0 \leq s \leq t$. Thus the second order moments exist for $Z_t^{(N)}$ and the two integrals in (1.7). Using the inequality $(a+b)^2 \leq 2\left(a^2+b^2\right)$, the Cauchy-Schwarz inequality, we obtain from

$$\begin{aligned} E\left(\left|Z_t^{(N)}\right|^2\right) \leq & 2E\left(\left|\int_0^t I_s^{(N)}\left(a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right)\right) ds\right|^2\right) \\ & + 2E\left(\left|\int_0^t I_s^{(N)}\left(b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right) dW_s\right|^2\right) \\ \leq & 2T\int_{t_0}^t E\left(\left|I_s^{(N)}\left(a\left(s, X_s\right) - a\left(s, \tilde{X}_s\right)\right)\right|^2\right) ds \\ & + 2\int_0^t E\left(\left|I_s^{(N)}\left(b\left(s, X_s\right) - b\left(s, \tilde{X}_s\right)\right)\right|^2\right) ds \end{aligned}$$

which we combine with (1.8) to get

$$E\left(\left|Z_t^{(N)}\right|^2\right) \leq L\int_0^t E\left(\left|Z_s^{(N)}\right|^2\right) ds$$

for $t \in [0, T]$ where $L = 2\left(T+1\right)K^2$. We then apply the Gronwall inequality to conclude that

$$E\left(\left|Z_t^{(N)}\right|^2\right) = E\left(\left|I_t^{(N)}\left(X_t - \tilde{X}_t\right)\right|^2\right) = 0$$

and hence that $I_t^{(N)} X_t = I_t^{(N)} \tilde{X}_t$, w.p.1, for each $t \in [0, T]$. Note,

$$P\left(I_t^{(N)} \not\equiv 1 \forall t \in [0, T]\right) \leq P\left(\sup_{0\leq t\leq T}\left|X_t\right| > N\right) + P\left(\sup_{0\leq t\leq T}\left|\tilde{X}_t\right| > N\right).$$

Since the sample paths are continuous almost surely they are bounded almost surely, we can make the probability on the right arbitrarily small by taking N sufficiently large. This means that $P\left(X_t \neq \tilde{X}_t\right) = 0$ for each $t \in [0, T]$.

And then $P\left(X_t \neq \tilde{X}_t : t \in D\right) = 0$ for any **countably** dense subset D of $[0, T]$. As the solutions are continuous and coincide on a countably dense subset of $[0, T]$, they must coincide, almost surely, on the entire interval $[0, T]$.

1.4 Existence

Picard iteration:

$$X_t^{n+1} = X_0 + \int_0^t a(s, X_s^n)ds + \int_0^t b(s, X_s^n)dW_s, \tag{1.9}$$

X_t^n all measurable in A_t, $X_t^0 = X_0$. Second moment estimate:

$$\begin{aligned} E[|X_t^{n+1}|^2] &\leq 3E[|X_0|^2] + 3E[|\int_0^t a(s, X_s^n)ds|^2] \\ &+ 3E[|\int_0^t b(s, W_s)dW_s|^2] \\ &\leq 3E[|X_0|^2] + 3TE[\int_0^t |a(s, X_s^n)|^2 ds] \\ &\quad +3E[\int_0^t |b(s, X_s^n)|^2 ds] \\ &\leq 3E[|X_0|^2] + 3(T+1)K^2 E[\int_0^t 1 + |X_s^n|^2] \end{aligned} \tag{1.10}$$

implying $\sup_{t\in[0,T]} E[|X_t^n|^2] \leq C_0 < \infty$, for all n. (by induction)
As in uniqueness:

$$E(|X_t^{n+1} - X_t^n|^2) \leq L \int_0^t E(|X_s^n - X_s^{n-1}|^2)ds \tag{1.11}$$

iterating:

$$E(|X_t^{n+1} - X_t^n|^2) \leq \frac{L^n}{(n-1)!} \int_0^t (t-s)^{n-1} E[|X_s^1 - X_s^0|^2]ds. \tag{1.12}$$

By growth bound (A2):

$$E[|X_t^1 - X_t^0|^2] \leq L \int_0^t (1 + E[|X_s^0|^2]) \leq C_1 < \infty, \tag{1.13}$$

then (1.12):

$$E(|X_t^{n+1} - X_t^n|^2) \leq C_1 L^n t^n / n!,$$

or:

$$\sup_{t\in[0,T]} E(|X_t^{n+1} - X_t^n|^2) \leq C_1 L^n T^n / n!,$$

implying X_t^n converges in mean square sense for $t \in [0, T]$.

Note: $\mathcal{L}_T^2$ equipped with,

$$\|f\|_{2,T} = \sqrt{\int_0^T E\left(f(t,\cdot)^2\right) dt} \tag{1.14}$$

is a Banach space space.

The limit is a solution of SIE (some work needed to achieve path-wise convergence and then pass the limit, see KL's book p133).

Prior estimate If $E[|X_0|^{2n}] < \infty$,

$$E[|X_t|^{2n}] \le (1 + E[|X_0|^{2n}])e^{Ct}. \tag{1.15}$$

Eg of Non existence Consider SDE we discuss last lecture:

$$dY_t = -\frac{1}{2}e^{-2Y_t}dt + e^{-Y_t}dW_t, \tag{1.16}$$

has unique solution:

$$Y_t = \ln(W_t + e^{Y_0}),$$

valid until time:

$$T = T(Y_0(\omega)) = \min\{t \ge 0 : W_t(\omega) + e^{Y_0(\omega)} = 0.\}$$

Growth condition (A2) is not satisfied.

1.5 Generalized results

1.5.1 Lipschitz Condition

Lipschitz condition in b can be replaced by Yamada condition:

$$|b(t,x) - b(t,y)| \le \rho(|x-y|),\ \rho(0) = 0,\ \int_{0^+} \rho^{-2}(u)du = +\infty. \tag{1.17}$$

One can take $\rho(u) = u^q$, $q \in [1/2, 1]$.
Example:

SDE $dX_t = |X_t|^q dW_t$, $X_0 = 0$, has unique solution if $q \in [1/2, 1]$.

But SDE:

$$dX_t = \frac{1}{3}X_t^{1/3}dt + X_t^{2/3}dW_t, \tag{1.18}$$

with initial data $X_0 = 0$ has nontrivial solution (hence nonuniqueness):

$$X_t = (W_t/3)^3.$$

1.5.2 Growth Condition

Growth condition on drift coefficient a can be replaced by:

$$xa(t,x) \le K(1 + |x|^2),$$

allowing $a = x - x^3$.

The idea comes from

$$\frac{1}{2}\frac{d}{dt}x^2 = xa(x) = x^2 - x^4 \le x^2$$

1.6 Stability

If as $\epsilon \to 0$:

$$E[|X_0^\epsilon - X_0|^2] \to 0,$$

$$\sup_{|x|\le N} |a^\epsilon(t,x) - a(t,x)| \to 0,\ t \in [0,T],\ \forall N,$$

$$\sup_{|x|\le N} |b^\epsilon(t,x) - b(t,x)| \to 0,\ t \in [0,T],\ \forall N,$$

then (by Gronwall Inequality):

$$\sup_{t\in[0,T]} E[|X_t^\epsilon - X_t|^2] \to 0.$$

Example: Solutions of SDE:

$$dX_t^\epsilon = a(t, X_t^\epsilon)dt + \epsilon dW_t, \tag{1.19}$$

converge in the mean square sense to those of deterministic ODE: $X_t' = a(t, X_t)$ with same initial data.

2 Diffusion Process and Weak Solution

2.1 Strong Solution as a diffusion process

Recall definition of Diffusion process,
Markov process with transition density is called *diffusion process* if the following limits exist:
Jump:

$$\lim_{t\to s^+} \frac{1}{t-s} \int_{|y-x|>\epsilon} p(s,x;t,y)dy = 0,$$

Drift:

$$\lim_{t\to s^+} \frac{1}{t-s} \int_{|y-x|\le\epsilon} (y-x)p(s,x;t,y)dy = a(s,x),$$

Diffusion:

$$\lim_{t\to s^+} \frac{1}{t-s} \int_{|y-x|\le\epsilon} (y-x)^2\, p(s,x;t,y)dy = b^2(s,x).$$

Theorem Assume that a and b are continuous and that A1-A3 hold. Then the solution X_t of (1.1) for any fixed initial value X_{t_0} is a diffusion process on $[t_0, T]$ with drift $a(t,x)$ and diffusion coefficient $b(t,x)$.

2.2 Diffusion process as a Weak Solution

Note as we turn investigate strong solution as a diffusion process, we only verify the transition density of the distribution.

Given a diffusion process, Y, on $[0, T]$ with drift $a(t, y)$ and strictly positive diffusion coefficient $b(t, y)$. Under some assumptions (see Theorem 4.7.1 in K-L's book), we can show Y_t satisfies some SDE driven by BM.

To show this, we define functions g and $\bar{a}$ by

$$g(t, y) = \int_0^y \frac{dx}{b(t, x)} \tag{2.20}$$

and

$$\bar{a}(t, z) = \left(\frac{\partial g}{\partial t} + a\frac{\partial g}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 g}{\partial y^2}\right)\left(t, g^{-1}(t, z)\right) \tag{2.21}$$

with a and b evaluated at (t, y), where $y = g^{-1}(t, z)$ is the inverse of $z = g(t, y)$ Then we define a process $Z_t = g\left(t, Y_t\right)$, which is a diffusion process with drift $\bar{a}(t, z)$ and diffusion coefficient 1.

And the process

$$\tilde{W}_t = Z_t - Z_0 - \int_0^t \bar{a}\left(s, Z_s\right) ds \tag{2.22}$$

which will turn out to be a Wiener process. (Some proof needed here.)

Consequently (2.22) will be equivalent to the stochastic differential equation

$$dZ_t = \bar{a}\left(t, Z_t\right) dt + 1 d\tilde{W}_t$$

which, by (2.20),(2.21) and Ito's formula, will imply that Y_t is a solution of the stochastic differential equation

$$dY_t = a\left(t, Y_t\right) dt + b\left(t, Y_t\right) d\tilde{W}_t.$$

Example to discuss

1. In vector SDE
$$dX_t = I_2 dW_t$$
X^1, X^2 are not identical in path but share same distribution.

2. $$dX_t = \operatorname{sgn} X_t dt + dW_t$$
where $\operatorname{sgn} x = +1$ if $x \geq 0$ and -1 if $x < 0$, only has weak solutions, but no strong solution for the initial value $X_0 = 0$. In fact, if X_t is such a weak solution for the Wiener process W, then $-X_t$ is a weak solution for the Wiener process $-W$. These solutions have the same probability law, but not the same sample paths.